A Note on Multivariate Limits

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Abstract

In this note we define functions with geometrically intuitive graphs whose limit along a prescribed polynomial path has value zero while the limit along every other polynomial path has value one.

When teaching multi-variable calculus, we do our best to convince students that the notion of a limit of a function of several variables is much stronger than that of limits of a function of a single variable. To that end, we provide examples of functions whose limiting value depends upon the straight line path that passes through the point in question. Then we inevitably haul out an old chestnut like the function $f(x, y) = \dfrac{2x^2y}{x^4 + y^2}$ whose limit is 0 along every line through the origin, but which has value 1 along the curve $y = x^2$.

While presenting such an example, a student asked what the graphs of such functions looked like. Obviously one can draw the graph of the above function, but the question begged for a more complete answer, especially in terms of how natural the graph of such functions can be.

Seeking examples of functions analogous to f, having an isolated discontinuity and limits with constant value along lines, but a distinct value along some prescribed curve, would lead to complicated graphs. Instead, if one chooses to introduce functions with many (but natural) discontinuities, the examples become trivial to envision: a high plateau with a river running through it.

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Theorem. *Let $p(x)$ be a polynomial of degree $n > 1$ with $p(0) = 0$. Define the function $f(x, y)$ by*

$$f(x,y) = \begin{cases} 0 & \text{if } y = p(x) \\ 1 & \text{if } y \neq p(x) \end{cases}.$$

Then the limit of f (as $(x, y) \to (0, 0)$) along $y = p(x)$ is zero, while the limit along any line through the origin is one. Moreover, if q is any non-constant polynomial passing through the origin with $q \neq p$, then the limit of f along the graph of $q(x)$ also has value one.

Proof. That the limit along $y = p(x)$ is zero is obvious. We separate the proof into two parts, lines through the origin and polynomials q of degree at least two. This is done not only since calculus students will understand the proof for lines, but also since the line $x = 0$ is not the graph of a (polynomial) function.

The essential claim is that there is a disk D centered at the origin in which the graphs of $y = p(x)$ and of a given line or the graph of $y = q(x)$ do not intersect except at the origin. Given such a D, it is clear that on $D \setminus (0, 0)$, f has value one, establishing the theorem.

To see the claim, note that the graph of $y = p(x)$ crosses the line $x = 0$ precisely once as p is a function. Any other line can be represented as $y = kx$ for some k, and equating $p(x) = kx$ yields a polynomial equation of degree n, and hence yielding at most $n - 1$ points of intersection other than $(0, 0)$. Choosing a disk centered at the origin small enough to exclude these points has the desired property.

In the case of a general polynomial q of degree m, Bezout's [1] theorem guarantees at most mn points of intersection, so once again we may choose a disk with the desired property. $\square$

References

[1] W. Fulton, *Algebraic Curves*, W. A. Benjamin, Inc. (1969).

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