The Arithmetic and Combinatorics of Buildings for Sp_n

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Abstract

In this paper we investigate both arithmetic and combinatorial aspects of buildings and associated Hecke operators for $Sp_n(K)$ with K a local field. We characterize the action of the affine Weyl group in terms of a symplectic basis for an apartment, characterize the special vertices as those which are "self-dual" with respect to the induced inner product, and finally establish a one-to-one correspondence between the special vertices in an apartment and the elements of the quotient $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \ldots, 1)$.

We then give a natural representation of the local Hecke algebra over $\mathbb{Q}_p$ acting on the special vertices of the Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$. Finally, we give an application of the Hecke operators defined on the building by characterizing minimal walks on the building for Sp_n.

1 Introduction

Buildings play a large role in the study of classical groups, and in particular a role in the study of Hecke algebras associated to these groups. In [5], Serre defined Hecke operators acting on trees (the building associated to SL_2 over a local field), and this work was generalized to SL_n in [2]. In this paper we investigate both arithmetic and combinatorial aspects of buildings and associated Hecke operators for $Sp_n(K)$ with K a local field.

Compared to the theory of buildings for the special linear group, the theory for the symplectic group is far less developed, so the first part of this paper is devoted to giving more concrete characterizations of the vertices in an apartment with particular attention to so-called special vertices. We note that in the case of SL_n all vertices in the building are special. We characterize the action of the affine Weyl group in terms of a symplectic basis for an apartment, characterize the special vertices as those which are "self-dual" with respect to the induced inner product, and finally establish a one-to-one correspondence between the special vertices in an apartment and the elements of the quotient $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \ldots, 1)$.

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We next establish connections between the symplectic elementary divisor theory of lattices over the ring of integers $\mathcal{O}$ of K and double cosets of the group $\Gamma = Sp_n(\mathcal{O})$. Using this correspondence, we define Hecke operators on the building which act as generalized adjacency operators on the underlying graph. We then give a natural (essentially faithful) representation of the local Hecke algebra over $\mathbb{Q}_p$ acting on the special vertices of the Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$. Finally, we give an application of the Hecke operators defined on the building by characterizing minimal walks on (the one-subcomplex of) the building for Sp_n.

2 The building for Sp_n

In this section we consider the building for Sp_n over a local field, and give intrinsic characterizations of its apartments and special vertices. In particular, we give a concrete characterization of the action of the affine Weyl group, $\widetilde{C}_n$, in terms of a symplectic basis for an apartment. Moreover, after associating each vertex with a homothety class of a lattice in the symplectic space, we show that special vertices are precisely those in the building which are "self-dual" with respect to the induced inner product. For our application to walks on the building, we further establish a one-to-one correspondence between the special vertices in an apartment and the elements of $\mathbb{Z}^{n+1}/\mathbb{Z}(2,1,\dots 1)$. The induced group structure will be interpreted in terms of special vertices in the section on walks on the building.

Throughout, let K be a local field, $\mathcal{O}$ its ring of integers, $\pi \in \mathcal{O}$ a uniformizing parameter, $k = \mathcal{O}/\pi\mathcal{O}$ the residue field, and $(V, \langle *, * \rangle)$ be a symplectic (non-degenerate alternating) space of dimension $2n$ over K. Define the group of symplectic similitudes of K by

$$\begin{aligned} GSp_n(K) &= \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n}(K) \mid A^tC = C^tA,\ B^tD = D^tB,\ A^tD - C^tB = r(M)I_{2n}\} \\ &= \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n}(K) \mid AB^t = B^tA,\ CD^t = DC^t,\ AD^t - BC^t = r(M)I_{2n}\} \end{aligned}$$

where $r(M) \in K^\times$. Now let $S = K^\times/\mathcal{O}^\times$; for convenience, we take $S = \{\pi^\nu \mid \nu \in \mathbb{Z}\}$. We will denote by $GSp_n^S(K) = \{M \in GSp_n(K) \mid r(M) \in S\}$. It is useful to note that $Sp_n(K) \subset GSp_n^S(K)$. Finally, let $\Gamma = Sp_n(\mathcal{O})$.

The Bruhat-Tits building for $Sp_n(K)$ is an n-dimensional simplicial complex, Δ_n, whose vertices are homothety classes of lattices in V. One defines an incidence relation on the vertices, and the resulting flag complex is the building. Generally our focus will be on an apartment in the building for which we need a careful understanding of how the vertices are indexed by classes of lattices. Some of the basic material can be found in Chapter 20 of [3], which we supplement where germane.

Definition 2.1. *An $\mathcal{O}$-lattice $\Lambda \subset V$ is a free $\mathcal{O}$-module of rank n, and is called primitive if $\langle \Lambda, \Lambda \rangle \subseteq \mathcal{O}$ and $\langle *, * \rangle$ induces a non-degenerate form on the alternating space $\Lambda/\pi\Lambda$ over k.*

Following [3], we first give a general description of the building. We describe an apartment system for the building as follows (see [3]). A *frame* is an unordered n-tuple $\{\lambda_1^1, \lambda_1^2\}, \dots, \{\lambda_n^1, \lambda_n^2\}$ of pairs of lines $\{\lambda_i^1, \lambda_i^2\}$ so that $V = \sum_1^n (\lambda_i^1 + \lambda_i^2)$, $(\lambda_i^1 + \lambda_i^2)$ is orthogonal to $(\lambda_j^1 + \lambda_j^2)$ for $i \neq j$, and each $(\lambda_i^1 + \lambda_i^2)$ is a hyperbolic plane. We say that the frame determines the apartment Σ. Vertices in Σ are homothety classes of lattices, denoted $[\Lambda]$. A vertex $[\Lambda]$ lies in Σ (determined by the above frame), if there are free $\mathcal{O}$-modules $M_i^j \subset \lambda_i^j$ so that $\Lambda = \oplus_{i,j} M_i^j$ for some (and hence every) representative Λ of the homothety class. More concretely, vertices of the building are homothety classes of lattices $[\Lambda]$ which possess a representative Λ such that: there exists a lattice Λ_0 with $\pi^{-1}\Lambda_0$ primitive, $\Lambda_0 \subseteq \Lambda \subseteq \pi^{-1}\Lambda_0$, and $\langle \Lambda, \Lambda \rangle \subseteq \pi\mathcal{O}$; or equivalently, Λ/Λ_0 is a totally isotropic k-subspace of the non-degenerate alternating space $\pi^{-1}\Lambda_0/\Lambda_0$. Now to define the building, we start with the set of vertices, and define an incidence relation on them as follows: For vertices t, t', we say $t \sim t'$ if there are lattices $\Lambda_t \in t$ and $\Lambda_{t'} \in t'$ and a lattice Λ_0 such that $\pi^{-1}\Lambda_0$ is primitive, $\Lambda_0 \subseteq \Lambda_t \subseteq \pi^{-1}\Lambda_0$, $\pi\Lambda_0 \subseteq \Lambda_{t'} \subseteq \pi^{-1}\Lambda_0$, and either $\Lambda_t \subset \Lambda_{t'}$ or $\Lambda_{t'} \subset \Lambda_t$. The associated flag complex yields the building. The maximal simplices (chambers) are unordered $(n+1)$-tuples $[\Lambda_0], [\Lambda_1] \dots, [\Lambda_n]$ of homothety classes of lattices with representatives Λ_i satisfying: $\pi^{-1}\Lambda_0$ is primitive, $\Lambda_0 \subseteq \Lambda_i \subseteq \pi^{-1}\Lambda_0$, and $\Lambda_1/\Lambda_0 \subset \Lambda_2/\Lambda_0 \subset \cdots \subset \Lambda_n/\Lambda_0$ is a maximal isotropic flag of k-subspaces in $\pi^{-1}\Lambda_0/\Lambda_0$.

Now we establish a more concrete realization of the apartment. Fix a symplectic basis $\{u_1, w_1, \dots, u_n, w_n\}$ of V ($\langle u_i, w_i \rangle = 1$, $\langle u_i, u_i \rangle = \langle w_i, w_i \rangle = 0$), and let Λ be the $\mathcal{O}$-lattice $\Lambda = \mathcal{O}\pi^{a_1}u_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}u_n \oplus \mathcal{O}\pi^{b_1}w_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}w_n$. With the basis fixed, we often denote this lattice as $(\pi^{a_1}, \dots \pi^{a_n}; \pi^{b_1}, \dots, \pi^{b_n})$. We note that $\langle \Lambda, \Lambda \rangle \subseteq \mathcal{O}$ iff $\langle \pi^{a_i}u_i, \pi^{b_i}w_i \rangle = \pi^{a_i+b_i} \in \mathcal{O}$ which is true iff $a_i + b_i \geq 0$. Given $a_i + b_i \geq 0$, the induced alternating form on $\Lambda/\pi\Lambda$ is non-degenerate over $k = \mathcal{O}/\pi\mathcal{O}$ iff $a_i + b_i = 0$ for all i.

Example 2.2. *Let $\{u_1, \dots, u_n, w_1, \dots, w_n\}$ be a symplectic basis for V (with $\langle u_i, w_i \rangle = 1$), and put $\lambda_i^1 = Ku_i$ and $\lambda_i^2 = Kw_i$. The frame $\{\lambda_i^1, \lambda_i^2\}$ defines an apartment Σ. Let $\Lambda_0 = \pi(\oplus \mathcal{O}u_i \oplus \mathcal{O}w_i)$. Then $\pi^{-1}\Lambda_0$ is primitive. Denote by $[\pi^{a_1}, \dots, \pi^{a_n}; \pi^{b_1}, \dots, \pi^{b_n}]$ the class of the lattice $\mathcal{O}\pi^{a_1}u_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}u_n \oplus \mathcal{O}\pi^{b_1}w_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}w_n$. Then the following flags determine (fundamental) chambers in Σ.*

$[\Lambda_0] = [\pi, \dots, \pi; \pi, \dots, \pi] \subset [\Lambda_1] = [1, \pi, \dots, \pi; \pi, \dots, \pi] \subset [\Lambda_2] = [1, 1, \pi, \dots, \pi; \pi, \dots, \pi] \subset \cdots \subset [\Lambda_n] = [1, 1, \dots, 1; \pi, \dots, \pi]$.

$[\Lambda_0] = [\pi, \dots, \pi; \pi, \dots, \pi] \subset [\Lambda_1] = [\pi, \dots, \pi; 1, \pi, \dots, \pi] \subset [\Lambda_2] = [\pi, \dots, \pi; 1, 1, \pi \dots, \pi] \subset \cdots \subset [\Lambda_n] = [\pi, \dots, \pi; 1, 1, \dots, 1]$.

To see how the rest of the apartment is laid out, one must understand the action on the lattices of the reflections which generate the Weyl group associated to the Bruhat-Tits building for $Sp_n(K)$. The affine Weyl group is of type $\widetilde{C}_n$ which has Coxeter diagram:

with $(n+1)$ "vertices", and the two endpoints being "special" vertices in the sense of [4]. The Coxeter diagram for C_n is the same with the last special vertex (and associated "edge") deleted. Associated to each "vertex" i in the Coxeter diagram is a reflection s_i, and the collection of reflections satisfy the standard rules $s_i^2 = 1$ and $s_i s_j$ has order m_{ij}, indicated by the Coxeter diagram ($m_{12} = m_{n(n+1)} = 4$, $m_{i(i+1)} = 3, i \neq 1, n$, and $m_{ij} = 2$ otherwise).

Acting on the symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$, define the reflections (any basis vector not specified is fixed):

- s_1: Interchange u_1 and w_1
- s_j $(2 \le j \le n)$: Interchange $u_{j-1} \leftrightarrow u_j$ and $w_{j-1} \leftrightarrow w_j$
- s_{n+1}: $u_n \mapsto \pi^{-1} w_n$, $w_n \mapsto \pi u_n$

That is, acting on a vertex $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$,

- s_1 takes $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$ to $[\pi^{b_1}, \pi^{a_2}, \ldots, \pi^{a_n}; \pi^{a_1}, \pi^{b_2} \ldots, \pi^{b_n}]$
- s_j $(2 \le j \le n)$: takes $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$ to $[\pi^{a_1}, \ldots, \pi^{a_j}, \pi^{a_{j-1}}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_j}, \pi^{b_{j-1}}, \ldots, \pi^{b_n}]$
- s_{n+1}: takes $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$ to $[\pi^{a_1}, \ldots, \pi^{a_{n-1}}, \pi^{b_n+1}; \pi^{b_1}, \ldots, \pi^{b_{n-1}}, \pi^{a_n-1}]$

The group $\widetilde{C}_n$ is generated by $s_1, \ldots, s_{n+1}$ subject only to the relations given by the Coxeter data above.

Now we proceed to label the apartment Σ. Each chamber in the building contains two special vertices. In the case of a fundamental chamber, one of them is the vertex fixed by the reflections $s_1, \ldots, s_n$, and the other is fixed by $s_2, \ldots, s_{n+1}$. Label the special vertex fixed by $s_1, \ldots, s_n$ as $v_0 = [\pi, \ldots, \pi; \pi, \ldots, \pi]$. From above, we see that the vertex $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$ is fixed by s_j $(2 \le j \le n)$ iff $a_{j-1} = a_j$ and $b_{j-1} = b_j$. The vertex is fixed by s_{n+1} iff $a_n = b_n + 1$. Thus $a_i = b_i + 1$ for all i, and so the vertex $v_1 = [\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}] = [\pi, \ldots, \pi; 1, \ldots, 1]$ is another special vertex in a fundamental chamber. Now fix a fundamental chamber containing these two vertices. Then the codimension-one faces of this fundamental chamber may be labeled by the reflections s_i so as to generate the rest of the apartment. In labeling the apartment, it is useful to first establish the "residue" of the vertex v_0 (that is the set of chambers in Σ containing it). This is simply obtained by letting the spherical Weyl group $C_n = \langle s_1, \ldots, s_n \rangle \subsetneq \widetilde{C}_n$ act on the fundamental chamber.

Example 2.3. *We illustrate this in the case of $n = 2$. Since Sp_n is of type C_n, the Weyl group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ (the signed permutation group) and has order $2^n n!$, so for $n = 2$ we expect 8 chambers in the apartment containing the given vertex $v_0 = [\Lambda_0]$, with $\Lambda_0 = \pi(\mathcal{O}u_1 \oplus \mathcal{O}u_2 \oplus \mathcal{O}w_1 \oplus \mathcal{O}w_2)$. Thus we start with a fundamental chamber given by the flag $\Lambda_0 = (\pi, \pi, \pi.\pi) \subset (1, \pi, \pi, \pi) \subset (1, 1, \pi, \pi) \subset (1, 1, 1, 1) = \pi^{-1}\Lambda_0$, and act on this flag with the group $C_2 = \langle s_1, s_2 \rangle = \{1, s_1, s_2, s_2 s_1, s_1 s_2, s_1 s_2 s_1, s_2 s_1 s_2, s_2 s_1 s_2 s_1 = s_1 s_2 s_1 s_2\}$.*

The pairs of vertices defining the chambers containing v_0 are:

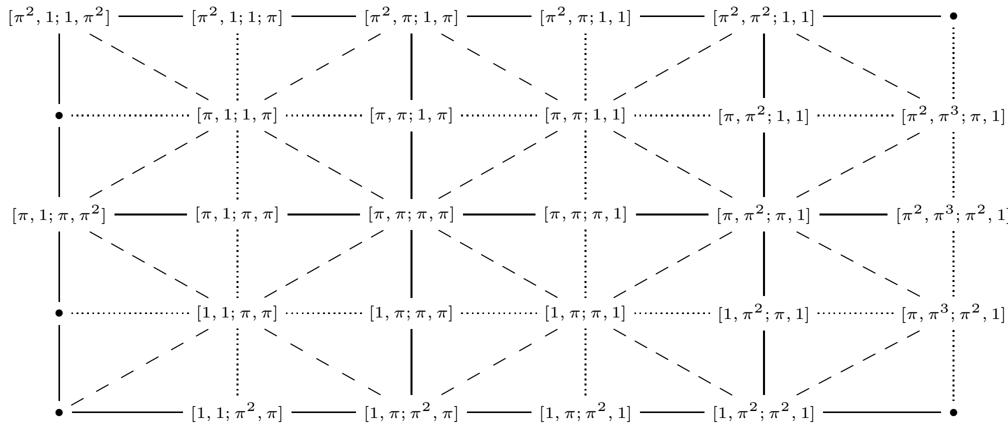
$$[1, \pi, \pi, \pi] \subset [1, 1, \pi, \pi], \qquad [\pi, \pi, 1, \pi] \subset [\pi, 1, 1, \pi], \qquad [\pi, 1, \pi, \pi] \subset [1, 1, \pi, \pi],$$

$$[\pi, \pi, \pi, 1] \subset [1, \pi, \pi, 1], \qquad [\pi, 1, \pi, \pi] \subset [\pi, 1, 1, \pi], \qquad [\pi, \pi, \pi, 1] \subset [\pi, \pi, 1, 1],$$

$$[1, \pi, \pi, \pi] \subset [1, \pi, \pi, 1], \qquad [\pi, \pi, 1, \pi] \subset [\pi, \pi, 1, 1].$$

Given the residue of a vertex, we continue the labelling of the apartment by use of the full affine group $\widetilde{C}_n$.

Example 2.4. *For $Sp_2(K)$ we have the following (partial) labeling of an apartment by classes of lattices:*



At this point we give a lattice-theoretic characterization of the special vertices. From the general theory of buildings we know that $Sp_n(K)$ acts transitively on the chambers of the building, so in particular maps our fundamental chamber to any other chamber, and does so in a "type preserving" manner. In particular, special vertices are mapped to special vertices. Also note that in the fundamental apartment, the special vertex $v_0 = [\pi, \ldots, \pi; \pi, \ldots, \pi]$ is mapped to the other special vertex $v_1 = [\pi, \ldots, \pi; 1, \ldots, 1]$ by means of the matrix $\mathrm{diag}(1, \ldots, 1, \pi^{-1}, \ldots, \pi^{-1}) \in GSp_n^S(K)$. Thus it is clear that every special vertex in the building is the image of v_0 under the action of $GSp_n^S(K)$. The converse is also true; to see this, we give an alternate characterization of special vertices as those which are "self-dual".

Let $\Lambda = \mathcal{O}\pi^{a_1}u_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}u_n \oplus \mathcal{O}\pi^{b_1}w_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}w_n$. The dual lattice $\Lambda^\sharp$ is defined to be $\{v \in V \mid \langle v, \Lambda \rangle \subseteq \mathcal{O}\}$. It too is a lattice, and it it easily seen from the bilinearity of the alternating form that $\Lambda^\sharp = \mathcal{O}\pi^{-b_1}u_1 \oplus \cdots \oplus \mathcal{O}\pi^{-b_n}u_n \oplus \mathcal{O}\pi^{-a_1}w_1 \oplus \cdots \oplus \mathcal{O}\pi^{-a_n}w_n$. It is also clear that $(\pi^\nu \Lambda)^\sharp = \pi^{-\nu}\Lambda^\sharp$, so $[\Lambda^\sharp]$ depends only on $[\Lambda]$, and in particular $[\Lambda] = [\Lambda^\sharp]$ iff $\pi^\mu \Lambda^\sharp = \Lambda$ for some $\mu \in \mathbb{Z}$.

Proposition 2.5. *Let $\Lambda = \mathcal{O}\pi^{a_1}u_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}u_n \oplus \mathcal{O}\pi^{b_1}w_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}w_n$. Then $[\Lambda] = [\Lambda^\sharp]$ iff there exists an integer μ, so that for all i, $a_i + b_i = \mu$. In this case we call the vertex self-dual.*

Proof. Using our explicit characterization of the dual lattice, $[\Lambda] = [\Lambda^\sharp]$ iff there exists an integer μ so that $\pi^\mu \Lambda^\sharp = \Lambda$, that is (by comparing coefficients of the u_i and w_i) iff $\pi^\mu \pi^{-b_i} = \pi^{a_i}$ and $\pi^\mu \pi^{-a_i} = \pi^{b_i}$ which is iff $\mu = a_i + b_i$ for all i. ☐

Proposition 2.6. *If $\Lambda = \mathcal{O}\pi^{a_1}u_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}u_n \oplus \mathcal{O}\pi^{b_1}w_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}w_n$, and the vertex $[\Lambda]$ is self-dual, then it image under the affine Weyl group $\widetilde{C}_n$ is again a self-dual vertex. Moreover, the image of any non-self dual vertex is again not self-dual.*

Proof. We need only check this for the generators s_i of the affine Weyl group, and all of these assertions are obvious from the definitions above. $\square$

Remark 2.7. *From the above it now follows that $GSp_n^S(K)$ acts transitively on the special vertices in the building. In particular, we have already observed that every special vertex can be obtained by acting on the vertex $v_0 = [\pi, \ldots, \pi; \pi \ldots, \pi]$ in the fundamental apartment Σ by an element of $GSp_n^S(K)$. We need only observe that the action of $GSp_n^S(K)$ on v_0 is always a special vertex. To see this, recall that $Sp_n(K)$ acts in a type-preserving manner on vertices, and in particular takes special vertices to special vertices. Since $\Gamma = Sp_n(\mathcal{O}) \subset Sp_n(K)$, we know that any element $\xi \in GSp_n^S(K)$ will act in the same way as any element of $\Gamma\xi\Gamma$. Thus by Lemma 3.1 (see below), we may assume that $\xi = \mathrm{diag}(\pi^{a_1}, \ldots, \pi^{a_n}, \pi^{b_1}, \ldots, \pi^{b_n})$ with $a_i + b_i$ constant. Finally, it is clear that the action of this ξ on v_0 produces a self-dual vertex in Σ, which by Proposition 2.6 must be special.*

For our application to walks on the building, it is convenient here to make one further characterization of the special vertices in an apartment. As above, we work in a fixed apartment with symplectic basis $\{u_i, w_i\}$. From the discussion above, we saw that a vertex $v_0 = [\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$ represented by the lattice Let $\Lambda = \mathcal{O}\pi^{a_1}u_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}u_n \oplus \mathcal{O}\pi^{b_1}w_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}w_n$ is special (self-dual) if and only if $a_i + b_i = \mu$ is constant. Moreover, the lattice Λ is completely characterized by the data $(\mu, a_1, \ldots, a_n) \in \mathbb{Z}^{n+1}$. For two special vertices v_0 and $v_0' = [\pi^{a_1'}, \ldots, \pi^{a_n'}; \pi^{b_1'}, \ldots, \pi^{b_n'}]$, we have that $v_0 = v_0'$ iff $a_i' = a_i + k$ and $b_i' = b_i + k$ for some $k \in \mathbb{Z}$. Denoting v_0 by $[(\mu, a_1, \ldots, a_n)]$ we see that $[(\mu, a_1, \ldots, a_n)] = [(\mu', a_1', \ldots, a_n')]$ iff $a_i' = a_i + k$ and $\mu' = \mu + 2k$. Thus there is a one-to-one correspondence between the special vertices in the apartment and the elements of the quotient $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \ldots, 1)$. We exploit this identification and the inherited group structure in the final section of the paper where we characterize minimal walks on the building.

3 A representation of the local Hecke algebra

The goal of this section is to define an essentially faithful representation of a local Hecke algebra acting on the special vertices of the building for Sp_n. The representation is quite natural, generalizing both the notion of adjacency operators on a graph and Serre's action of the Hecke algebra on trees (see [5] for the case of SL_2, and [2] for higher rank generalizations). To do so, we need to discuss how the lattices which define the special vertices of the building are connected to elementary divisors, and in turn how the elementary divisors are connected to double cosets of the Hecke algebra.

3.1 Symplectic lattices and elementary divisors

We begin with a short discussion about lattices and elementary divisors in the symplectic setting. Generalizing (ever so slightly) the context of the previous section, let E be a global or local field, $\mathcal{O}$ its ring of integers, and $(V, \langle *, * \rangle)$ a $2n$-dimensional symplectic space over E. To study the symplectic divisor theory (elementary divisors with respect to the symplectic group), we assume that $\mathcal{O}$ is a PID (e.g., if E is any local field or a global field of class number one), and let $S = E^\times/\mathcal{O}^\times$. For $E = \mathbb{Q}$, we let $S = \mathbb{Q}_+^\times$, the positive rationals, while for $E = \mathbb{Q}_p$, we let $S = \{p^\nu \mid \nu \in \mathbb{Z}\}$. As before, we denote by $GSp_n^S(E) = \{M \in GSp_n(E) \mid r(M) \in S\}$, so when $E = \mathbb{Q}$, $GSp_n^S(E)$ is the classical group of similitudes $GSp_n^+(\mathbb{Q})$. We again note that $Sp_n(E) \subset GSp_n^S(E)$, and put $\Gamma = Sp_n(\mathcal{O})$.

Fix a symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ of V satisfying $\langle u_i, w_j\rangle = \delta_{ij}$ (Kronecker delta), $\langle u_i, u_j\rangle = \langle w_i, w_j\rangle = 0$. With obvious modification to the proof, the following is Lemma 3.6 of [1].

Lemma 3.1. *Let $\xi \in GSp_n^S(E)$. Then every double coset $\Gamma\xi\Gamma$ has a unique representative of the form $sd(\xi) = \operatorname{diag}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ where $\alpha_i, \beta_i \in S$ and $\alpha_i \mid \alpha_{i+1}$, $\alpha_n \mid \beta_n$, $\beta_{i+1} \mid \beta_i$, and $\alpha_i\beta_i = r(\xi)$*

We call a lattice symplectic if it has an $\mathcal{O}$-basis which is a symplectic basis for V with respect to the alternating bilinear form on V. The following proposition is easily established.

Proposition 3.2. *Let $\mathcal{L}$ be a symplectic lattice. Then $\Gamma = Sp_n(\mathcal{O})$ can be identified with $\{A \in GSp_n^S(E) \mid \mathcal{L}A = \mathcal{L}\}$, where A acts on $\mathcal{L}$ as the matrix of a linear transformation with respect to a fixed basis of $\mathcal{L}$.*

To set up the correct analog of elementary divisor theory, we need to fuss a bit more than in the general linear case. To begin, fix a symplectic lattice $\mathcal{L}$ and put $\mathcal{R} = \mathcal{R}_{\mathcal{L}} = \{\mathcal{L}A \mid A \in GSp_n^S(E)\}$. Note that in the general linear case, GSp_n would be replaced by GL_{2n}, and $\mathcal{R}$ would be the set of all lattices of full rank in V, and so $\mathcal{R}$ would not need to be defined at all.

Lemma 3.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be lattices in $\mathcal{R}$. Then there exists a symplectic basis $\{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ of V, and elements $\alpha_i, \beta_i \in S$ satisfying $\beta_1\mathcal{O} \subset \cdots \subset \beta_n\mathcal{O} \subset \alpha_n\mathcal{O} \subset \cdots \subset \alpha_1\mathcal{O}$ and $\beta_i\alpha_i = r \in S$ such that $\mathcal{M} = \bigoplus_{i=1}^n \mathcal{O}u_i \oplus \bigoplus_{i=1}^n \mathcal{O}w_i$ and $\mathcal{N} = \bigoplus_{i=1}^n \mathcal{O}\alpha_i u_i \oplus \bigoplus_{i=1}^n \mathcal{O}\beta_i w_i$.*

Remark 3.4. *The ideals $\alpha_i\mathcal{O}$ and $\beta_j\mathcal{O}$ are called the symplectic divisors of $\mathcal{N}$ in $\mathcal{M}$, and coincide with the standard elementary divisors $\{\mathcal{M} : \mathcal{N}\}$ since $\Gamma \subset SL_{2n}(\mathcal{O})$. That is, if we choose two lattices from $\mathcal{R}$ and consider their elementary divisors in the traditional sense, they are in fact symplectic elementary divisors with the additional properties stated above. In particular, if $\mathcal{M}$ and $\mathcal{N}$ are as in the lemma, we will write $\{\mathcal{M} : \mathcal{N}\} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$ to mean there exist bases of $\mathcal{M}$ and $\mathcal{N}$ as in the lemma.*

Proof. Since $\mathcal{M}$ and $\mathcal{N}$ are in $\mathcal{R}$, there exists an $A \in GSp_n^S(E)$ with $\mathcal{N} = \mathcal{M}A$. Assume that Γ is identified with the stabilizer of $\mathcal{M}$. By Lemma 3.1, $sd(A) = \operatorname{diag}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) = \gamma_1 A\gamma_2$ for some $\gamma_i \in \Gamma$, where $sd(A)$ is the "symplectic divisor" matrix of A. Finally, it is clear that since $\mathcal{M}\gamma_i = \mathcal{M}$, that $\{\mathcal{M} : \mathcal{N}\} = \{\mathcal{M}\gamma_1 : \mathcal{M}\gamma_1 A\} = \{\mathcal{M}\gamma_1\gamma_2 : \mathcal{M}\gamma_1 A\gamma_2\} = \{\mathcal{M} : \mathcal{M}sd(A)\} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$, from which the lemma follows. □

Lemma 3.5. *For A and B in $GSp_n^S(E)$, $\Gamma A = \Gamma B$ if and only if $\mathcal{L}A = \mathcal{L}B$.*

Proof. $\Gamma A = \Gamma B$ if and only if $AB^{-1} \in \Gamma$, which by Proposition 3.2 is true if and only if $\mathcal{L} = \mathcal{L}AB^{-1}$. ◻

Lemma 3.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be lattices in $\mathcal{R}$. The elementary divisors of $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{L}$ satisfy $\{\mathcal{L} : \mathcal{M}\} = \{\mathcal{L} : \mathcal{N}\}$ if and only if there exists an $A \in \Gamma$ such that $\mathcal{M}A = \mathcal{N}$.*

Proof. The result is clear if there exists an $A \in \Gamma$ such that $\mathcal{M}A = \mathcal{N}$. To prove the converse, we note that by definition of the symplectic elementary divisors, there exist elements α_i, $\beta_i \in S$ satisfying $\beta_1\mathcal{O} \subset \cdots \subset \beta_n\mathcal{O} \subset \alpha_n\mathcal{O} \subset \cdots \subset \alpha_1\mathcal{O}$ and $\beta_i\alpha_i = r \in S$ and symplectic $\mathcal{O}$-bases

$$\{u_1^{(j)}, \ldots, u_n^{(j)}; w_1^{(j)}, \ldots, w_n^{(j)}\} \qquad (j = 1, 2)$$

of $\mathcal{L}$ such that

$$\begin{aligned} \mathcal{L} &= \bigoplus_{i=1}^n \mathcal{O}u_i^{(1)} \oplus \bigoplus_{i=1}^n \mathcal{O}w_i^{(1)} & \mathcal{M} &= \bigoplus_{i=1}^n \mathcal{O}\alpha_i u_i^{(1)} \oplus \bigoplus_{i=1}^n \mathcal{O}\beta_i w_i^{(1)} \\ \mathcal{L} &= \bigoplus_{i=1}^n \mathcal{O}u_i^{(2)} \oplus \bigoplus_{i=1}^n \mathcal{O}w_i^{(2)} & \mathcal{N} &= \bigoplus_{i=1}^n \mathcal{O}\alpha_i u_i^{(2)} \oplus \bigoplus_{i=1}^n \mathcal{O}\beta_i w_i^{(2)} \end{aligned}$$

Let A be the matrix of the linear transformation (with respect to either basis) taking $u_i^{(1)} \mapsto u_i^{(2)}$, and $w_i^{(1)} \mapsto w_i^{(2)}$. Clearly $A \in Sp_n(E) \subset GSp_n^S(E)$ as it maps one symplectic basis to another. Since $\mathcal{L}A = \mathcal{L}$, $A \in \Gamma$ by Proposition 3.2 above. Since A obviously maps $\mathcal{M}$ to $\mathcal{N}$, the proof is complete. ◻

Proposition 3.7. *Let $\mathcal{L} \in \mathcal{R}$, Γ the stabilizer of $\mathcal{L}$ as above, $A \in GSp_n^S(E)$, and*

$$\Gamma A \Gamma = \Gamma sd(A)\Gamma = \Gamma \operatorname{diag}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)\Gamma.$$

Then $\Gamma\xi \mapsto \mathcal{L}\xi$ gives a one-to-one correspondence between the cosets $\Gamma\xi$ in $\Gamma A\Gamma$ and lattices $M \in \mathcal{R}$ with $\{\mathcal{L} : \mathcal{M}\} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots \beta_n\}$.

Proof. We may assume that $A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$. If $\Gamma\xi = \Gamma A\delta$ with $\delta \in \Gamma$, then $\mathcal{L}\xi \in \mathcal{R}$ and we have $\{\mathcal{L} : \mathcal{L}\xi\} = \{\mathcal{L} : \mathcal{L}A\delta\} = \{\mathcal{L} : \mathcal{L}A\} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots \beta_n\}$. Conversely, if $\mathcal{M} \in \mathcal{R}$ and $\{\mathcal{L} : \mathcal{M}\} = \{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots \beta_n\}$, then by Lemma 3.6 there exists an element $B \in \Gamma$ such that $\mathcal{M} = \mathcal{L}AB$. Clearly $\Gamma AB \subset \Gamma A\Gamma$. The correspondence is one-to-one since by Lemma 3.5, $\Gamma\xi = \Gamma\zeta$ if and only if $\mathcal{L}\xi = \mathcal{L}\zeta$. ◻

3.2 The representation

We now give a natural representation of the local Hecke algebra in which the Hecke operators act on the (special) vertices of the building for $Sp_n(K)$. In addition, we shall show how the operators in this representation space correspond to "adjacency operators" on the underlying 1-complex of the building. In the next section, we use these operators to characterize minimal walks on the building.

It is well-known (see [1]) that the "local" Hecke algebras characterize the global Hecke algebra, but since we don't require any of the global theory for this application, we shall simply define the local Hecke algebra, $\mathcal{H}_p$. For concreteness, we work over the local field $K = \mathbb{Q}_p$, and continue the notation of the earlier sections with $\mathcal{O} = \mathbb{Z}_p$, $\pi = p$, $S = \{p^\nu \mid \nu \in \mathbb{Z}\}$, and $\Gamma = Sp_n(\mathbb{Z}_p)$. To be precise, the algebra we define, $\mathcal{H}_p$, is isomorphic to the localization of the global Hecke algebra, but in particular, it is not the entire Hecke ring associated to the Hecke pair $Sp_n(\mathbb{Z}_p)$ and $GSp_n(\mathbb{Q}_p)$. Rather it is the subalgebra (over $\mathbb{Q}$) generated by double cosets $\Gamma\xi\Gamma$ with $\xi \in GSp_n^S(\mathbb{Q}_p)$. By Lemma 3.1, we may assume all ξ have the form $\xi = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n}, p^{b_1}, \ldots, p^{b_n})$, where $a_1 \leq \cdots \leq a_n \leq b_n \leq \cdots \leq b_1$. To introduce the algebra structure on $\mathcal{H}_p$, we give its multiplication law (e.g., see section 3.1 of [6]):

Let $\xi_1 = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n}, p^{b_1}, \ldots, p^{b_n})$, $\xi_2 = \mathrm{diag}(p^{c_1}, \ldots, p^{c_n}, p^{d_1}, \ldots, p^{d_n})$ be elements of $GSp_n^S(\mathbb{Q}_p)$ and write $\Gamma\xi_1\Gamma$ as the disjoint union $\cup\Gamma\alpha_i$, and write $\Gamma\xi_2\Gamma$ as the disjoint union $\cup\Gamma\beta_j$.

$$(\Gamma\xi_1\Gamma)(\Gamma\xi_2\Gamma) = \Gamma\xi_1\Gamma\xi_2\Gamma = \sum_{\Gamma\xi\Gamma} c(\xi)\Gamma\xi\Gamma$$

where the sum is over all double cosets $\Gamma\xi\Gamma \subset \Gamma\xi_1\Gamma\xi_2\Gamma$, and where $c(\xi)$ is the number of pairs (i, j) for which $\Gamma\alpha_i\beta_j = \Gamma\xi$.

We remark that they key to establishing the isomorphism between the localization of the global Hecke algebra and the local algebra $\mathcal{H}_p$ we have just defined is the following lemma.

Lemma 3.8. *Let $\Lambda = Sp_n(\mathbb{Z})$ and $\xi \in GSp_n^+(\mathbb{Q}) \cap GL_{2n}(\mathbb{Z}[p^{-1}])$. If $\Lambda\xi\Lambda$ is the disjoint union $\cup\Lambda\xi_i$, then $\Gamma\xi\Gamma$ is the disjoint union $\cup\Gamma\xi_i$.*

Proof. Without loss of generality, we may assume $\xi \in M_{2n}(\mathbb{Z})$ since the general case follows by multiplying by a power of p. First, it is clear that $\cup\Gamma\xi_i \subseteq \Gamma\xi\Gamma$, and that the ξ_i have integer entries. Next, we see that the cosets $\Gamma\xi_i$ are disjoint since if not then $\xi_i\xi_j^{-1} \in \Gamma \cap GL_{2n}(\mathbb{Z}[p^{-1}]) \subseteq \Lambda$. To show that the union is all of $\Gamma\xi\Gamma$, we need only show that $\xi\Gamma \subseteq \Gamma\xi\Lambda$, for then any element $\tilde{\gamma}_1\xi\tilde{\gamma}_2 = \tilde{\gamma}_3\xi\gamma$ ($\tilde{\gamma}_i \in \Gamma$, $\gamma \in \Lambda$), so $\xi\gamma \in \Lambda\xi_i$ for some i, hence $\tilde{\gamma}_1\xi\tilde{\gamma}_2 \in \Gamma\xi_i$. To see $\xi\Gamma \subseteq \Gamma\xi\Lambda$ is really just a density argument: Let $q = r(\xi)$ be the similitude factor associated to ξ. Recall (see [1]) that there is a natural surjective homomorphism $Sp_n(R) \to Sp_n(R/qR)$ with $R = \mathbb{Z}$ or $\mathbb{Z}_p$. Denote by $\Gamma(q)$ the kernel of the homomorphism $Sp_n(\mathbb{Z}_p) \to Sp_n(\mathbb{Z}_p/q\mathbb{Z}_p)$. From Chapter 2, §3.3 of [1], we have that $\Gamma(q) \subset \Gamma \cap \xi^{-1}\Gamma\xi$. Using the fact that $Sp_n(\mathbb{Z}_p/q\mathbb{Z}_p) \cong Sp_n(\mathbb{Z}/q\mathbb{Z})$ (and that $Sp_n(\mathbb{Z}) \to Sp_n(\mathbb{Z}/q\mathbb{Z})$ is surjective), we may write $\Gamma = \cup\Gamma(q)\delta_i$ with the $\delta_i \in \Lambda$. Let $\tilde{\gamma} \in \Gamma$, and write $\tilde{\gamma} = \tilde{\gamma}_0\delta_k$ for some $\tilde{\gamma}_0 \in \Gamma(q)$ and some k. Then

$$\xi\tilde{\gamma} = \xi\tilde{\gamma}_0\delta_k = \xi\tilde{\gamma}_0\delta_k(\delta_k^{-1}\xi^{-1})\xi\delta_k = (\xi\tilde{\gamma}_0\xi^{-1})\xi\delta_k \in \Gamma\xi\Lambda$$

since $\xi\tilde{\gamma}_0\xi^{-1} \in \xi\Gamma(q)\xi^{-1} \subset \xi(\Gamma \cap \xi^{-1}\Gamma\xi)\xi^{-1} \subset \Gamma$.

□

Finally, we are ready to define our representation of the local Hecke algebra $\mathcal{H}_p$ acting on the Bruhat-Tits building, Δ_n, for $Sp_n(\mathbb{Q}_p)$. We have previously noted that the vertices of the building, $\mathrm{Vert}(\Delta_n)$, are in one-to-one correspondence with homothety classes of lattices in our fixed $\mathbb{Q}_p$-vector space V, however our action will only be on the special vertices. So we let $\mathcal{B}$ be the rational vector space with basis consisting of the special vertices in $\mathrm{Vert}(\Delta_n)$.

Let L be a lattice in V with $[L]$ a special vertex in Δ_n, and identify $\Gamma = Sp_n(\mathbb{Z}_p)$ with the stabilizer of L in $GSp_n^S(\mathbb{Q}_p)$. Let $\xi = \mathrm{diag}(p^{a_1}, \dots, p^{a_n}; p^{b_1}, \dots, p^{b_n}) \in GSp_n^S(\mathbb{Q}_p)$. By Proposition 3.7, we know that the double coset $\Gamma\xi\Gamma$ determines a collection of right cosets $\{\Gamma\xi_\nu\}$ which are in one-to-one correspondence with the collection of lattices $\{M\}$ with $\{L : M\} = \{p^{a_1}, \dots, p^{a_n}; p^{b_1}, \dots, p^{b_n}\}$. Note that all of these lattices M are contained in $\mathcal{R} = \{LA \mid A \in GSp_n^S(\mathbb{Q}_p)\}$, and hence by the discussion above, their classes are all special vertices.

In the context of Hecke operators acting on modular forms, the natural action of a double coset on the modular form is to sum the actions on the form by the right cosets comprising the double coset. Using the notation above, it is then natural to define the operator $T_{\mathcal{B}}(p^{a_1}, \dots, p^{a_n}; p^{b_1}, \dots, p^{b_n}) \in \mathrm{End}(\mathcal{B})$ induced by:

$$T_{\mathcal{B}}(p^{a_1}, \dots, p^{a_n}; p^{b_1}, \dots, p^{b_n})([L]) = \sum_{\{L:M\}=\{p^{a_1},\dots,p^{a_n};p^{b_1},\dots,p^{b_n}\}} [M]$$

where the sum is over all (special) vertices in the building with prescribed elementary divisors. For brevity, we shall write $T_{\mathcal{B}}(\xi)([L]) = \sum_{\{L:M\}=\xi}[M]$. The map is clearly well-defined and (by definition) linear.

Theorem 3.9. *The correspondence $\Gamma\xi\Gamma \mapsto T_{\mathcal{B}}(\xi)$ induces a representation $\Psi : \mathcal{H}_p \to \mathrm{End}(\mathcal{B})$, whose kernel consists of double cosets of the form $\Gamma\xi\Gamma$ with $\xi = p^\mu I_{2n}$, $\mu \in \mathbb{Z}$.*

Proof. We first verify that Ψ is a ring homomorphism. Using the notation above, we have

$$\begin{aligned} T_{\mathcal{B}}(\xi_1)T_{\mathcal{B}}(\xi_2)([L]) &= T_{\mathcal{B}}(\xi_1)\Big(\sum_{\{L:M\}=\xi_2}[M]\Big) \\ &= \sum_{\{L:M\}=\xi_2}\ \sum_{\{M:N\}=\xi_1}[N] \end{aligned}$$

By Proposition 3.7, each lattice M for which $\{L : M\} = \xi_2$ is of the form $M = L\beta_j$. Now

$$\{M : N\} = \xi_1 \iff \{L\beta_j : N\} = \xi_1 \iff \{L : N\beta_j^{-1}\} = \xi_1$$

Now let P be such that $\{L : P\} = \xi_1$. Then again by Proposition 3.7, $P = L\alpha_i$ for some i. But then $P = N\beta_j^{-1}$, so $N = P\beta_j = L\alpha_i\beta_j$.

Thus, $T_{\mathcal{B}}(\xi_1)T_{\mathcal{B}}(\xi_2)([L]) = \sum_{\{L:M\}=\xi_2}\ \sum_{\{M:N\}=\xi_1}[N] = \sum_{i,j}[L\alpha_i\beta_j]$. From the discussion preceding the theorem (and once again Proposition 3.7), this last sum is exactly $\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)([L])$ which is the image of $(\Gamma\xi_1\Gamma)(\Gamma\xi_2\Gamma)$.

To compute the kernel of Ψ, suppose $\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)$ is the trivial map. Then

$$\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)([L]) = \sum_{\Gamma\xi\Gamma} c(\xi) \sum_{\{L:M\}=\xi} [M] = [L]$$

for all special vertices $[L] \in \mathrm{Vert}(\Delta_n)$. But the special vertices $[M] \in \mathrm{Vert}(\Delta_n)$ are a basis for $\mathcal{B}$, we have all the only one ξ, and for that ξ, $c(\xi) = 1$. Thus we have $\sum_{\{L:M\}=\xi}[M] = [L]$ for all $[L]$. Now if $\Gamma\xi\Gamma = \cup\Gamma\xi_\nu$, then by Proposition 3.7, $\sum_{\{L:M\}=\xi}[M] = \sum_\nu [L\xi_\nu] = [L]$, so there can be only one right coset: $\Gamma\xi\Gamma = \Gamma\xi$, and $[L\xi] = [L]$. Since $\{L : L\xi\} = \xi$, we must have $\xi = p^\mu I_{2n}$ for some integer μ. □

We have suggested that this operator is natural by means of its analogy with summing right cosets of a double coset, but in fact it also intrinsic in terms of the underlying structure of the building Δ_n, where the operators can, at least at the level of apartments in the building, be identified with adjacency operators. Since this is just motivation, we content ourselves to looking at the generators for the local Hecke algebra for Sp_2 for which we have provided pictures of the apartment.

Example 3.10. *For Sp_2, there are three generators of the algebra $\mathcal{H}_p$, $T(p) = \Gamma\,\mathrm{diag}(1, 1, p, p)\Gamma$, $T_1^2(p^2) = \Gamma\,\mathrm{diag}(1, p, p^2, p)\Gamma$ and $T_2^2(p^2) = \Gamma\,\mathrm{diag}(p, p, p, p)\Gamma$ whose images under the representation are respectively $T_{\mathcal{B}}(1, 1, p, p)$, $T_{\mathcal{B}}(1, p, p^2, p)$, and $T_{\mathcal{B}}(p, p, p, p)$. The last operator acts trivially, but the first two are of real interest. At least restricted to the fundamental apartment (see Example 2.4), we see that $T_{\mathcal{B}}(1, 1, p, p)$ sums the four special vertices closest to $[p, p; p, p]$, namely $[1, 1; p, p] + [1, p; p, 1] + [p, p; 1, 1] + [p, 1; 1, p]$ while $T_{\mathcal{B}}(1, p, p^2, p)$ sums the four special vertices "next closest" to $[p, p; p, p]$, namely $[1, p^2; p, p] + [p, p^2; p, 1] + [p^2, p; 1, p] + [p, 1; p, p^2]$. Thus both operators act very much like adjacency operators on the underlying 1-complex.*

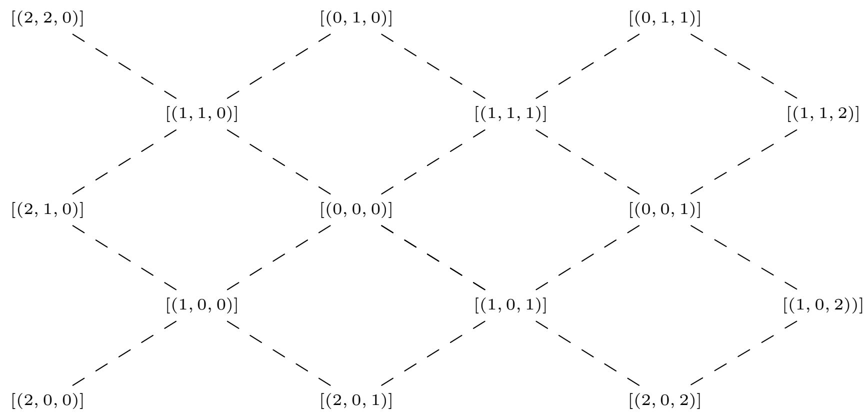
Finally, it is interesting to note what happens when we consider the sum of certain non-special vertices, for example consider the sum $T = [p, p; p, 1]+[p, p; 1, p]+[p, 1; p, p]+[1, p; p, p]$, consisting of the four non-special vertices closest to $[p, p, p, p]$. We observe that if we take $[L] + [M] = [L + M]$, then $[p, p; p, 1] + [p, p; 1, p] = [p, p, 1, 1]$ and $[p, 1; p, p] + [1, p; p, p] = [1, 1; p, p]$, while $[p, p; p, 1] + [p, 1; p, p] = [p, 1; p, 1]$ and $[p, p; 1, p] + [1, p; p, p] = [1, p; 1, p]$, so that $2T = T_{\mathcal{B}}(1, 1; p, p)$. Thus there appears to be no loss of generality by restricting adjacency operators to special vertices.

4 Hecke Operators and Walks

In this last section, we characterize minimal walks in the building of a prescribed length in terms of the action of the Hecke operators defined in the previous section.

Fix an apartment in the building by specifying a symplectic basis $\{u_1, w_1, \ldots, u_n, w_n\}$. We showed previously that the special vertices in the apartment are in one-to-one correspondence with the elements of $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, \ldots, 1)$.

Example 4.1. *For $Sp_2(K)$ we have the following (partial) labeling of the special vertices in an apartment by elements of $\mathbb{Z}^3/\mathbb{Z}(2,1,1)$. Note that in considering the 1-subcomplex of the apartment, we have removed all non-special vertices and the corresponding edges. Compare with Example 2.4.*



The natural group operation defined on $\mathbb{Z}^{n+1}/\mathbb{Z}(2,1,\dots,1)$ induces one the special vertices of an apartment. Moreover there is a natural geometric interpretation of this group operation as well. Consider the special vertices in the residue of a fundamental chamber containing $[(0,\dots,0)]$. Recalling that the spherical Weyl group, C_n, is isomorphic to the signed permutation group $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$, it is easy to see that the collection of special vertices in this residue (excluding $[(0,\dots,0)]$ itself) consists of all vertices of the form $[(1,\varepsilon)]$ where $\varepsilon \in \{0,1\}^n$. In Example 4.1 the special vertices are labeled counterclockwise $[(1,1,1)]$, $[(1,1,0)]$, $[(1,0,0)]$, and $[(1,0,1)]$, and they define "directions" in which to move (relative to $[(0,\dots,0])$ within the apartment which is consistent with the group law: For example, from $[(0,0,0)]$ moving 2 units in the direction indicated by $[(1,1,1)]$ and then one unit in the direction indicated by $[(1,0,1)]$ brings us to $[(3,2,3)] = [(1,1,2)]$. Thus we can think of a vertex $[(\mu,a_1,\dots,a_n)]$ as the endpoint of a walk along the 1-subcomplex of the apartment (consisting of only the special vertices and associated edges) which is given by moving a certain number of units in the directions mentioned above. Our goal is to characterize the endpoints of minimal walks on this graph.

Now the geometric action of our Hecke operator $T_{\mathcal{B}}$ becomes a bit clearer. Recall

$$T_{\mathcal{B}}(p^{a_1},\dots,p^{a_n};p^{b_1},\dots,p^{b_n})([L]) = \sum_{\{L:M\}=\{p^{a_1},\dots,p^{a_n};p^{b_1},\dots,p^{b_n}\}} [M]$$

Restricted to our given apartment, this sum is fairly easy to characterize. All lattices M in the apartment have the form $[p^{c_1},\dots,p^{c_n};p^{d_1},\dots,p^{d_n}]$. For simplicity, normalize $L = [p^0,\dots,p^0]$. Then $\{L:M\} = \{p^{a_1},\dots,p^{a_n};p^{b_1},\dots,p^{b_n}\}$ means that each c_i and d_i are chosen from among the a_i and b_i. But the choices are more constrained. For each i, c_i is either some $a_{\sigma(i)}$ or some $b_{\sigma(i)}$ for $\sigma \in S_n$. But then d_i is determined by the choice of c_i since $c_i + d_i$ is constant. In particular (assuming the normalization of L as above), the set of lattices M with the prescribed elementary divisors are those obtained by acting on $[p^{a_1}, \dots, p^{a_n}; p^{b_1}, \dots, p^{b_n}]$ by all the elements of the spherical Weyl group C_n.

The interpretation of $T_{\mathcal{B}}(p^{a_1}, \dots, p^{a_n}; p^{b_1}, \dots, p^{b_n})$ on the building Δ_n is a bit more complicated. By a minimal walk between two vertices, we simply mean a walk (a sequence of vertices $\{v_1, \dots, v_m\}$ each pair $\{v_i, v_{i+1}\}$ connected by an edge), between the two vertices which is of minimal length. Again we reiterate that we are considering only the 1-subcomplex of the building spanned by the special vertices. We characterize the endpoints of minimal walks in the building in the following theorem.

Theorem 4.2. *Let $v_0 = [L]$ be a special vertex in the Bruhat-Tits building Δ_n for $Sp_n(K)$ which is represented by the homothety class of the lattice L. The set of special vertices in the building which are endpoints of minimal walks of length m from v_0 is*

$$\sum_{0 \le a_2 \le \cdots \le a_n \le m/2} T_{\mathcal{B}}(1, p^{a_2}, \dots, p^{a_n}; p^m, p^{m-a_2}, \dots, p^{m-a_n})([L]).$$

Proof. Consider a minimal walk, γ, between two vertices v_0 and v_m in Δ_n. Denote the walk by the sequence of vertices through which it passes: $\gamma = \{v_0, v_1, \cdots, v_m\}$. Choose chambers C_0 and C_m with $v_0 \in C_0$ and $v_m \in C_m$, and let A be an apartment containing the chambers C_0 and C_m. Finally, let $\rho = \rho_{A,C_0}$ be the canonical retraction of Δ_n onto A centered at C_0.

Since the retraction ρ is a simplicial map, it takes the walk γ to another walk $\rho(\gamma) = \{\rho(v_0), \rho(v_1), \dots, \rho(v_{m-1}), \rho(v_m)\}$ contained in A. But v_0 and v_m are both in A, so are fixed pointwise by ρ, making $\rho(\gamma)$ a walk in A from v_0 to v_m. Moreover, it is clear that $\rho(\gamma)$ has length at most m, since it has at most $m+1$ distinct vertices defining the walk. Finally, since m is the length of any minimal walk from v_0 to v_m, we must have that $\rho(\gamma)$ has length m, and hence is a minimal walk in A from v_0 to v_m.

Since our interest is only to count the endpoints of minimal walks of length m, we may assume from the argument above that any such walk is wholly contained in an apartment. Thus we need only characterize the vertices of an apartment which are the endpoints of minimal walks (in that apartment) of length m. Let $v = [a_1, \dots, a_n; b_1, \dots, b_n]$ ($a_i + b_i = \mu$) be such a vertex. The Weyl group acting on the apartment will take any walk in the apartment to another of the same length. Since we will use the Weyl group to count endpoints of minimal walks in the apartment there is no loss of generality to assume that v is chosen with $0 \le a_1 \le \cdots \le a_n \le b_n \le \cdots \le b_1$. Moreover, since the vertex is defined by the homothety class of a lattice, we may assume that $a_1 = 0$. Recall that there is a one-to-one correspondence between the vertices of an apartment and elements in $\mathbb{Z}^{n+1}/\mathbb{Z}(2, 1, 1, \dots, 1)$. Our normalized v has the form $v = [(\mu, 0, a_2, \dots, a_n)]$, where $0 \le a_2 \le \cdots \le a_n \le \mu$. In fact all the $a_i \le \mu/2$ since $a_i \le a_n \le b_n$ and $a_n + b_n = \mu$. We claim that $\mu = m$. Define the elements of $\mathbb{Z}^{n+1}$: $\varepsilon_0 = (1, 0, \dots, 0)$, $\varepsilon_1 = (1, 0, \dots, 0, 1), \dots$, $\varepsilon_{n-1} = (1, 0, 1, \dots, 1)$. First note that the directions $[\varepsilon_0]$, $[\varepsilon_1], \dots$, $[\varepsilon_{n-1}]$ are independent in the sense that $\sum_{k=0}^{n-1} c_k \varepsilon_k \in \mathbb{Z}(2, 1, \dots, 1)$ iff $\sum_{k=0}^{n-1} c_k \varepsilon_k = 0$ iff $c_k = 0$ for all k. Now we return to our vertex $v = [(\mu, 0, a_2, \dots, a_n)]$ as above. If $\mu = 1$ then $0 \le a_2 \le \cdots \le a_n \le 1/2$, so $v = [(1, 0, \dots, 0)]$ is one of the special vertices in the residue of $(0, \dots, 0)]$, and hence the endpoint of a walk of length one.

Next consider the case $\mu > 1$. Then

$$v = [(\mu, 0, a_2, \ldots, a_n)] = a_2[\varepsilon_{n-1}] + (a_3 - a_2)[\varepsilon_{n-2}] + \cdots + (a_n - a_{n-1})[\varepsilon_1] + (\mu - a_n)[\varepsilon_0].$$

Each summand has the form $c[\varepsilon_i]$ and so represents a walk of length c in the direction $[\varepsilon_i]$. By the independence of the $[\varepsilon_i]$, we conclude the above walk is minimal (and of length μ) , hence $\mu = m$.

For a vertex v, denote by v^{C_n} the orbit of v under the action of the spherical Weyl group. Then in a given apartment, the endpoints of minimal walks of length m starting from $[0, 0, \ldots, 0]$ is given by

$$\sum_{0 \le a_2 \le \cdots \le a_n \le m/2} [(m, 0, a_2, \ldots, a_n)]^{C_n}.$$

From this, the theorem follows immediately. □

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