BRUHAT-TITS BUILDINGS AND HECKE OPERATORS

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Abstract. Serre introduced the notion of a Hecke operator acting on the vertices of a tree (the Bruhat-Tits building for $SL_2(\mathbb{Q}_p)$). In this paper we give a natural characterization of an algebra of Hecke operators acting on the vertices of the Bruhat-Tits building for $SL_n(\mathbb{Q}_p)$. This algebra of operators arises as a representation of an abstract Hecke algebra which, from work of Andrianov, is known to be isomorphic to the ring of symmetric polynomials in n variables. We show that this ring of symmetric polynomials acts in a completely natural way on the apartments of the building. Moreover, the abstract Hecke ring gives rise to n families of Hecke operators whose generating series produce "Euler factors" for various exterior powers. As a byproduct of our work, we give a natural characterization of Andrianov's spherical map by which he established this isomorphism between the abstract Hecke ring and the ring of symmetric polynomials.

1. Introduction

We are interested in characterizing an algebra of Hecke operators which act on the free abelian group generated by the vertices in the Bruhat-Tits building for $SL_n(\mathbb{Q}_p)$. This algebra of operators not only acts naturally in terms of the geometry of the building, but also is a representation space for the (p-part of the) standard Hecke algebra given in terms of double cosets. As a consequence, we obtain a geometrically motivated characterization of Andrianov's spherical map through which it is shown that the p-part of the Hecke algebra is isomorphic to the ring of symmetric polynomials $\mathbb{Q}[x_1, \dots, x_n]^{\text{sym}}$. Everything we do here will work just as well over any nonarchimedean local field, but for simplicity we restrict to $\mathbb{Q}_p$.

We begin with some notation concerning buildings and a summary of Serre's description of the Hecke operators when the building is a tree. Let V be an n-dimensional vector space over $\mathbb{Q}_p$, and Δ_n be the Bruhat-Tits building for $SL_n(\mathbb{Q}_p)$ (see [3] or [4]). The vertices of Δ_n are in one-to-one correspondence with homothety classes of lattices (free $\mathbb{Z}_p$-modules) in V of rank n. By the elementary divisor theorem, given two lattices $\mathcal{L}$ and $\mathcal{M}$, there exists a basis $\{e_1, \dots, e_n\}$ of V and rational integers $a_1, \dots, a_n$, so that

$$\mathcal{L} = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_n, \qquad \mathcal{M} = \mathbb{Z}_p p^{a_1} e_1 \oplus \cdots \oplus \mathbb{Z}_p p^{a_n} e_n.$$

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Given this notation, Serre [6], in the case of $n = 2$, defines the distance between the two vertices represented by the lattices $\mathcal{L}$ and $\mathcal{M}$ as $|a_1 - a_2|$. Generalizations for $n > 2$ appear in [5]. Using this distance function, Serre defines a graph by placing an edge between any two vertices which are unit distance apart. It then follows that the resulting graph (which is the Bruhat-Tits building for $SL_2(\mathbb{Q}_p)$) is a $(p+1)$-regular tree, and that vertices which are distance m from a given vertex are simply the endpoints of walks without backtracking of length m.

Serre defines a family of $\mathbb{Z}$-linear operators θ_ℓ which act on the free abelian group generated by the vertices of the tree by setting

$$\theta_\ell(v) = \sum_{d(v,w)=\ell} w,$$

that is, a vertex is mapped to the sum of its neighbors at distance ℓ. From the structure of the tree, he notes that

$$\theta_1\theta_1 = \theta_2 + (p+1)\theta_0 \quad \text{and} \quad \theta_1\theta_\ell = \theta_{\ell+1} + p\theta_{\ell-1} \text{ for } \ell \geq 2.$$

Compared to the recursions satisfied by classical Hecke operators $T(p^\ell)$ (or merely in terms of simplicity of expression) the recursion is slightly off in the base case. To correct this, Serre defines

$$T_0 = \theta_0, \quad T_1 = \theta_1, \quad \text{and} \quad T_\ell = \theta_\ell + T_{\ell-2} \text{ for } \ell \geq 2,$$

which yields (for all $\ell \geq 1$) the relation

$$T_1 T_\ell = T_{\ell+1} + pT_{\ell-1}.$$

While a generating series for the θ_ℓ operators is not quite as simple, the T_ℓ operators satisfy

$$\sum_{\ell \geq 0} T_\ell u^\ell = \left[1 - T_1 u + pu^2\right]^{-1}.$$

For obvious reasons, the operators T_ℓ are called Hecke operators. We observe that, while the Hecke operators satisfy a cleaner recursion than the θ_ℓs, their definition is in some ways less satisfying, since the θ_ℓs appear to be a more natural family of operators with respect to the structure of the tree.

In our generalizations to $SL_n(\mathbb{Q}_p)$ for $n > 2$, we provide motivation for why the operators T_ℓ are in fact quite natural.

2. Higher Rank Hecke Operators on Δ_n

With Serre's characterization of the Hecke operators on $SL_2(\mathbb{Q}_p)$ so readily in hand, we give the generalization to the case of $SL_n(\mathbb{Q}_p)$. While the definition given here is, in and of itself, a natural generalization of Serre's, we do not show until later how the corresponding algebra of these operators arises as the image of a representation of the abstract Hecke algebra given by double cosets. In later sections we also indicate a number of interesting "Euler" factors which arise by considering various generating series involving Hecke operators. On the other hand, we do take this opportunity to give an action of symmetric polynomials on the vertices of the building which will motivate the characterization of the isomorphism of the local Hecke algebra and the ring of symmetric polynomials established by Andrianov using the spherical map.

We begin with a little notation. As in the previous section, let V be an n-dimensional vector space over $\mathbb{Q}_p$, and let Δ_n be the Bruhat-Tits building for $SL_n(\mathbb{Q}_p)$. The vertices of Δ_n are in one-to-one correspondence with homothety classes of lattices in V of rank n, and we let $\mathcal{B}_0$ denote the free abelian group generated by the vertices of Δ_n and $\mathcal{B} = \mathbb{Q} \otimes \mathcal{B}_0$ the corresponding vector space over $\mathbb{Q}$.

For positive integers m and n, denote by $P_n(m)$ the set of partitions of m into n pieces such that a given partition satisfies: $m \geq i_1 \geq \cdots \geq i_n \geq 0$ (and $\sum i_k = m$). Endow $P_n(m)$ with the lexicographic ordering. Fix a lattice $\mathcal{L}$ in V whose homothety class $[\mathcal{L}]$ is a vertex in Δ_n, and let $\Gamma_p = GL_n(\mathbb{Z}_p)$.

With the lattice $\mathcal{L}$ fixed as above, we recall [7] that there is a $1-1$ correspondence between right cosets $\Gamma_p \xi_\nu$ in $\Gamma_p \operatorname{diag}(p^{a_1}, \ldots, p^{a_n})\Gamma_p$ and lattices $\mathcal{M}$ in $V = \mathcal{L} \otimes \mathbb{Q}_p$ with elementary divisors $\{\mathcal{L} : \mathcal{M}\} = \{p^{a_1}, \ldots, p^{a_n}\}$.

Thus, analogous to the action of a double coset on functions, it is natural to define the Hecke operator $T_B(p^{a_1}, \ldots, p^{a_n})$ (acting on $\mathcal{B}_0$ (or $\mathcal{B}$)) by

$$T_B(p^{a_1}, \ldots, p^{a_n})([\mathcal{L}]) = \sum_{\{\mathcal{L}:\mathcal{M}\} = \{p^{a_1}, \ldots, p^{a_n}\}} [\mathcal{M}].$$

Also with $\mathbf{i} = (i_1, \ldots, i_n) \in P_n(\ell)$, we define

$$T_B(p^\ell) = \sum_{\mathbf{i} \in P_n(\ell)} T_B(p^{i_1}, \ldots, p^{i_n}).$$

To foreshadow a bit, it is important to note that since the ring structure of the local Hecke algebra is characterized [7] precisely in terms of elementary divisors of lattices on V, we will have for free that the generating series for the operators $T_B(p^\ell)$ is expressible as the same rational function as the generating series for the classical Hecke operators $T(p^\ell)$ (see Theorem 3.21 of [7]), namely

$$\sum_{\ell \geq 0} T_B(p^\ell) u^\ell = \left[\sum_{k=0}^{n} (-1)^k p^{k(k-1)/2} \widetilde{t}_k^n(p) u^k \right]^{-1},$$

with the $\widetilde{t}_k^n(p)$ "elementary symmetric polynomials" defined in a subsequent section.

There is a corresponding definition of the Hecke operator on an apartment (in terms of classes of lattices):

$$T_A(p^{a_1}, \ldots, p^{a_n})([\mathcal{L}]) = {\sum_{\{\mathcal{L}:\mathcal{M}\} = \{p^{a_1}, \ldots, p^{a_n}\}}}^{*} [\mathcal{M}]$$

where the sum is over all lattices relative to the (unordered) basis $\{e_1, \ldots, e_n\}$ of $\mathcal{L}$. As we shall see, it is actually the Hecke operator restricted to the apartment which is the most natural from the point of view of symmetric polynomials, and represented our starting point in this investigation.

The operators defined above which act on the building are perfectly natural. Moreover, we show as an example, that in the case $n = 2$, the operators defined above are precisely the operators T_ℓ defined by Serre. The notation is as in the introduction.

Example 2.1. For $n = 2$, we have $T_B(p^\ell) = \sum_{k=0}^{[\ell/2]} T_B(p^k, p^{\ell-k})$.

Recall that Serre defined two operators

$$\theta_\ell([\mathcal{L}]) = \sum_{d([\mathcal{L}],[\mathcal{M}])=\ell} [\mathcal{M}],\ \ell \geq 1; \quad \theta_0 = 1$$
$$T_\ell([\mathcal{L}]) = \theta_\ell + T_{\ell-2},\ \ell \geq 2; \quad T_0 = \theta_0,\ T_1 = \theta_1$$

By Serre's definition of distance,

$$d([\mathcal{L}], [\mathcal{M}]) = \ell \text{ iff } \{\mathcal{L} : \mathcal{M}\} = \{p^a, p^b\} \text{ with } \ell = |a - b|$$

Since $[\mathcal{M}] = [p^k\mathcal{M}]$ we have that

$$\theta_\ell([\mathcal{L}]) = \sum_{\{\mathcal{L}:\mathcal{M}\}=\{1,p^\ell\}} [\mathcal{M}]$$

Claim: $T_\ell = T_B(p^\ell)$ for all $\ell \geq 0$. We proceed by induction on ℓ. For $\ell = 0$

this is trivial. For $\ell = 1$, we observe that $T_1([\mathcal{L}]) = \theta_1([\mathcal{L}]) = \sum_{\{\mathcal{L}:\mathcal{M}\}=\{1,p\}}[\mathcal{M}] = T_B(1, p)([\mathcal{L}]) = T_B(p)([\mathcal{L}])$.

Now assume that $\ell \geq 2$.

$$\begin{aligned}
T_\ell([\mathcal{L}]) &= \theta_\ell([\mathcal{L}]) + T_{\ell-2}([\mathcal{L}]) \\
&= \sum_{\{\mathcal{L}:\mathcal{M}\}=\{1,p^\ell\}} [\mathcal{M}] + T_B(p^{\ell-2})([\mathcal{L}]) \quad \text{by induction} \\
&= T_B(1,p^\ell)([\mathcal{L}]) + \sum_{k=0}^{[(\ell-2)/2]} T_B(p^k, p^{\ell-2-k})([\mathcal{L}]) \\
&= T_B(1,p^\ell)([\mathcal{L}]) + \sum_{k=0}^{[(\ell-2)/2]} T_B(p,p)T_B(p^k, p^{\ell-2-k})([\mathcal{L}]) \\
&= T_B(1,p^\ell)([\mathcal{L}]) + \sum_{k=0}^{[(\ell-2)/2]} T_B(p^{k+1}, p^{\ell-1-k})([\mathcal{L}]) \\
&= T_B(1,p^\ell)([\mathcal{L}]) + \sum_{k=1}^{[\ell/2]} T_B(p^k, p^{\ell-k})([\mathcal{L}]) \\
&= \sum_{k=0}^{[\ell/2]} T_B(p^k, p^{\ell-k})([\mathcal{L}]) = T_B(p^\ell)([\mathcal{L}]).
\end{aligned}$$

Now we introduce a labeling of the apartments in a building which will make natural an action of symmetric polynomials on the apartment, and hence provide the insight for connecting the local Hecke algebra with the ring of symmetric polynomials.

To specify an apartment in Δ_n, choose an unordered set of n one-dimensional subspaces $V_1, V_2, \ldots, V_n$, such that $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$. The vertices in the apartment can be viewed as homothety classes of lattices $L = \mathbb{Z}_p v_1 \oplus \cdots \oplus \mathbb{Z}_p v_n$ with $v_i \in V_i$ for each i. We denote by $\mathcal{A}_0$ the free abelian group generated by the vertices in this apartment, and $\mathcal{A}$ the corresponding vector space over $\mathbb{Q}$.

Now fix an ordering of the chosen subspaces V_i, which we will refer to as an *orientation* of the apartment. This orientation will allow us define first a labeling of the vertices in the apartment by monomials, and then a polynomial action on $\mathcal{A}$. To begin, fix a vertex v in the apartment, and let v correspond to the class of the lattice $L_0 = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_n$. Then the vertices of the apartment are in one-to-one correspondence with the classes of lattices $\mathbb{Z}_p p^{a_1} e_1 \oplus \cdots \oplus \mathbb{Z}_p p^{a_n} e_n$, where the a_i run over $\mathbb{Z}$. Focusing attention on the ordered n-tuple $(a_1, a_2, \ldots, a_n)$, the vertices of the apartment are thus in one-to-one correspondence with the elements of $\mathbb{Z}^n/\mathbb{Z} \cdot (1, 1, \ldots, 1)$.

Following [3], if $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are in $\mathbb{Z}^n$, write

$$u \preceq v \text{ if } u_i \leq v_i \leq u_i + 1 \text{ for all } i.$$

Calling two elements of $\mathbb{Z}^n/\mathbb{Z}\cdot(1,1,\dots,1)$ incident if they admit representatives u and v with $u \preceq v$ then produces a flag complex which defines the full simplicial structure of the apartment. It is clear that for every coset in $\mathbb{Z}^n/\mathbb{Z}\cdot(1,1,\dots,1)$, there is a unique representative $(a_1,\dots,a_n)$ in which all entries are nonnegative and at least one is zero. Using this representative, we label the corresponding vertex with the monomial $x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}$. As indicated by the structure of a flag complex, this labelling is far from arbitrary.

Consider our fixed vertex v represented by the lattice $L_0 = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_n$, and hence by the monomial 1. Any chamber (i.e., $(n-1)$-simplex) containing v corresponds to a maximal flag of lattices:

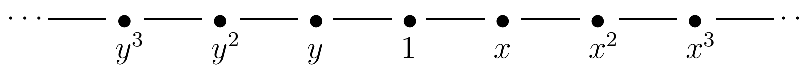
$$pL_0 = L_n \subsetneq L_{n-1} \subsetneq \cdots \subsetneq L_1 \subsetneq L_0.$$

Fix such a (fundamental) chamber, by choosing

$$L_i = \mathbb{Z}_p pe_1 \oplus \cdots \oplus \mathbb{Z}_p pe_i \oplus \mathbb{Z}_p e_{i+1} \oplus \cdots \oplus \mathbb{Z}_p e_n.$$

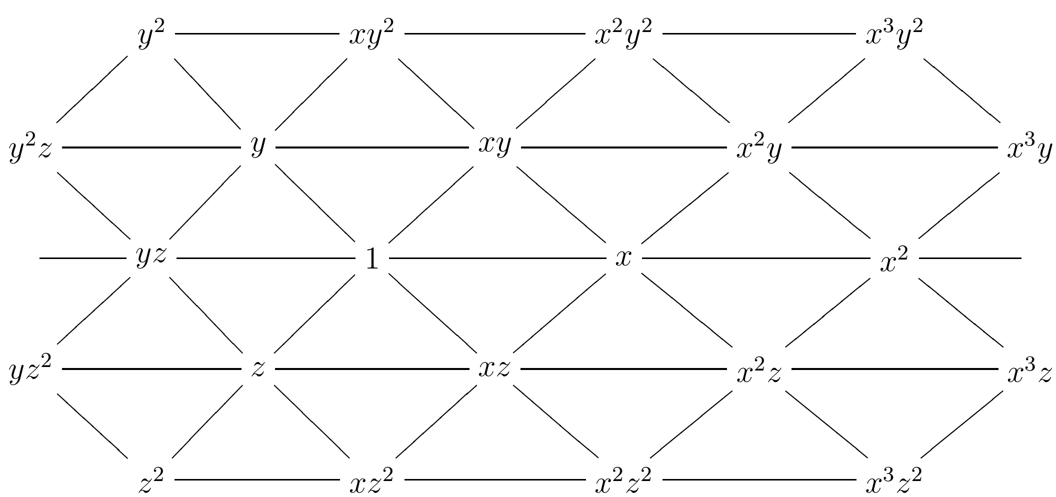
This chamber then has vertices labeled 1, x_1, x_1x_2, ... , $x_1x_2\cdots x_{n-1}$.

To illustrate the labeling, consider an apartment for SL_2:



We fix a vertex and label it 1, and then label a fundamental chamber (determining vertex x). Using the incidence relation defined above, it is clear that x is incident to 1, x^2 to x, x^3 to x^2, and so on. This labels the right hand side of the apartment. It is also clear that y is incident to 1, y^2 to y, and so on, which uniquely determines the left hand side.

Next consider a piece of a labeled apartment for SL_3:



As before, we fix a vertex and label it 1. Choosing a fundamental chamber fixes vertices labeled x and xy. It is clear that x and y are the only two vertices incident to both 1 and xy, hence the vertex labeled y is determined. z is also incident to 1, but not to y or x, which determines its placement. Loosely speaking, we see that there are three "principal directions" indicated by a motion from 1 to x, y or z. We label the vertices as follows: starting from 1 and moving in the x-direction, we label the vertices x, x^2, x^3, $\dots$. Similarly for movements in the y and z directions. The incidence relation says that the vertex xy is labeled as such because to get there from 1 we move one unit in the x direction and by one unit in the y direction.

For an apartment for SL_n, we choose an orientation and a vertex to be labeled 1, and thus a fundamental chamber in the manner described above. This has fixed the labelling of 1, x_1, x_1x_2, $\dots$, $x_1x_2\cdots x_{n-1}$. From here the rest of the labelling can be deduced from incidence relations. For consider the residue of the vertex 1, that is the collection of chambers which contain the vertex 1. It is well known that this is a spherical building of type A_n containing $n!$ chambers. Within the residue consider the $n-1$ chambers sharing a common codimension 1 face with the given chamber. Each such face is determined by $n-1$ vertices, and hence the nth vertex defining such a chamber is uniquely determined, and hence so is the class of the lattice which represents that vertex. Using the incidence relation, we may continue in this way to label the entire apartment.

Now we note that there is a natural action of monomials on the vertices of the apartment. For nonnegative integers a_i and indeterminates x_i, we say that a vertex w is a $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$-*translate* of v if there exists lattices $L_1 \in v$ and $L_2 \in w$ with $L_2 \subset L_1$ and $\{L_1 : L_2\} = \{p^{a_1}, p^{a_2} \dots , p^{a_n}\}$, where the latter notation represents the elementary divisors of L_2 relative to L_1. Note that since vertices are defined by classes of lattices, any vertex v is an $x_1\cdots x_n$ - translate of itself, i.e. the $x_1x_2\cdots x_n$ - translate of $v = [L]$ is $[pL] = v$. It is also obvious that the composition of translates corresponding to n-tuples of nonnegative integers $(a_1, a_2 \dots , a_n)$ and $(b_1, b_2 \dots , b_n)$ is equal to the translate corresponding to $(a_1 + b_1, a_2 + b_2, \dots , a_n + b_n)$, and that translates exist and are unique. However, if w is an $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$-translate of v, the n-tuple $(a_1, a_2, \dots , a_n)$ is determined only up to multiples of $(1, 1, \dots , 1)$.

Using the monomial labeling of the apartment, if w is an $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$-translate of v, then to go from v to w along the edges of the apartment, proceed a_i units in each of the x_i directions starting from v. Alternately the $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$-translate of the vertex labeled $x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n}$ is labeled by the product of these two monomials, modulo the relation $x_1x_2\cdots x_n = 1$.

This notion of translation extends to an action of the polynomial ring $\mathbb{Q}[x_1, x_2, \dots , x_n]$ on $\mathcal{A}$. Notice that this is independent of our choice of which vertex is labeled 1, since a change in that choice merely multiplies all labels by a fixed monomial. However, the action does depend on our choice of an orientation. We can however pass to a ring of orientation-independent operators by restricting to a smaller class of polynomials.

Since an orientation is defined by an ordering of the subspaces $V_1, V_2, \dots , V_n$, possible orientations correspond to elements of S_n. One readily sees that changing orientations affects our monomial labeling by $\sigma \in S_n$ acting on the variables $x_1, x_2, \dots , x_n$. Thus orientation-independent operators are simply symmetric polynomials, so the connection to Andrianov's work is becoming visible. To proceed, we need to work through some of the formalism of the local Hecke algebras.

3. Abstract Hecke Algebras

One goal of our work is to show that the p-part of the global Hecke algebra is naturally isomorphic to a ring of symmetric polynomials, and that the algebra of Hecke operators which act on the vertices of the Bruhat-Tits building is simply a representation of this standard Hecke algebra.

To begin, we review the forms of the Hecke theory given by Shimura [7] and by Andrianov and Zhuravlev [2], and then relate these to that of the Hecke algebra defined over a local field. Since these connections are mostly straightforward, detail will be minimized.

3.1. **Global Hecke Algebras.** The Hecke algebra described by Shimura [7] is given in terms of double cosets with respect to the discrete group $SL_n(\mathbb{Z})$, while the algebra described by Andrianov and Zhuravlev [2] is given in terms of formal sums of right cosets which are invariant under right action by the discrete group $GL_n(\mathbb{Z})$.

Let Γ be a subgroup of a group G and $S \supset \Gamma$, a semigroup contained in the commensurator of Γ in G. For $\xi \in S$, let

$$\Gamma\xi\Gamma = \cup \Gamma\xi_\nu$$

be the decomposition of the double coset into disjoint right cosets. Andrianov denotes by (ξ) the formal sum $\sum \Gamma\xi_\nu$. In Shimura's definition the Hecke algebra $H = H(\Gamma, S)$ is the free $\mathbb{Z}$-module generated by such $\Gamma\xi\Gamma$, while for Andrianov and Zhuravlev the generators are the (ξ). Since multiplication in the respective Hecke rings is defined in exactly the same way, obviously the map $\Gamma\xi\Gamma \mapsto (\xi)$ induces an isomorphism between the respective rings. We will therefore feel free to use either of the notations $\Gamma\xi\Gamma$ or (ξ) as is most convenient in what follows.

In [7] Shimura studies a Hecke ring H with respect to $\Gamma = SL_n(\mathbb{Z})$, and $S = M_n^+(\mathbb{Z})$ ($n \times n$ integral matrices with positive determinant), while Andrianov focuses on $\Gamma = GL_n(\mathbb{Z})$ and $S = GL_n(\mathbb{Q})$. However, the map $H(SL_n(\mathbb{Z}), M_n^+(\mathbb{Z})) \to H(GL_n(\mathbb{Z}), GL_n(\mathbb{Q}))$ defined by $(\xi) \mapsto (\xi)$ factors as

$$H(SL_n(\mathbb{Z}), M_n^+(\mathbb{Z})) \to H(SL_n(\mathbb{Z}), GL_n^+(\mathbb{Q})) \to H(GL_n(\mathbb{Z}), GL_n(\mathbb{Q})),$$

where the map on the left is an injection, and that on the right an isomorphism.

Draft Remark 1. The left hand map being an injection is clear.
For the right, we use Proposition III.1.9 of [2].

Lemma 3.1. *The rational Hecke algebra $R(SL_n(\mathbb{Z}), GL_n^+(\mathbb{Q}))$ is isomorphic to $R(GL_n(\mathbb{Z}), GL_n(\mathbb{Q}))$.*

Proof. In the notation of [2], let $\Gamma_0 = SL_n(\mathbb{Z})$, $\Gamma = GL_n(\mathbb{Z})$, $S_0 = GL_n^+(\mathbb{Q})$ and $S = GL_n(\mathbb{Q})$. We note that the conditions in equation (1.26) are satisfied:

1. $\Gamma_0 \subset \Gamma$
2. $S \subset \Gamma S_0$
3. $\Gamma \cap (S_0 \cdot S_0^{-1}) \subset \Gamma_0$

the last since the elements of $(S_0 \cdot S_0^{-1})$ have positive determinant.
Given a right coset Γs ($s \in S$), we have that $\Gamma s = \Gamma s_0$ for some $s_0 \in S_0$ by (2). It follows (see [2] that the map $\Gamma s \Gamma \mapsto \Gamma_0 s_0 \Gamma_0$ is a monomorphism of rings. Since $S_0 \subset S$, this map will be an isomorphism if we show that the degree of $\Gamma s_0 \Gamma$ is the same as the degree of $\Gamma_0 s_0 \Gamma_0$ for all $s_0 \in S_0$. That is, we must show that

$$[\Gamma : \Gamma \cap s_0^{-1}\Gamma s_0] = [\Gamma_0 : \Gamma_0 \cap s_0^{-1}\Gamma_0 s_0]$$

for all $s_0 \in S_0$.
Consider the map

$$\Gamma_0 \hookrightarrow \Gamma \to (\Gamma \cap s_0^{-1}\Gamma s_0)\backslash\Gamma \text{ given by } A \mapsto \left(\Gamma \cap s_0^{-1}\Gamma s_0\right) A$$

If A is in the kernel of this homomorphism, then $A \in \Gamma_0 \cap s_0^{-1}\Gamma s_0$, so $A = s_0 \gamma s_0^{-1}$ for some $\gamma \in \Gamma$. But then $\det(\gamma) = \det(A) = 1$, hence $\gamma \in \Gamma_0$ which yields that $A \in \Gamma_0 \cap s_0^{-1}\Gamma_0 s_0$. It follows immediately that the kernel is equal to $\Gamma_0 \cap s_0^{-1}\Gamma_0 s_0$, so we have an injection of $(\Gamma_0 \cap s_0^{-1}\Gamma_0 s_0)\backslash\Gamma_0$ into $(\Gamma \cap s_0^{-1}\Gamma s_0)\backslash\Gamma$.
To show surjectivity, let $B \in \Gamma$. We must show that there exists an $A \in \Gamma_0$ so that $(\Gamma \cap s_0^{-1}\Gamma s_0)B = (\Gamma \cap s_0^{-1}\Gamma s_0)A$. We are concerned with the degrees of the double cosets $\Gamma s_0 \Gamma$ and $\Gamma_0 s_0 \Gamma_0$, so without loss we may assume that s_0 is diagonal. Thus $s_0 \operatorname{diag}(-1, 1, \ldots, 1) s_0^{-1} = \operatorname{diag}(-1, 1, \ldots, 1)$, from which we deduce that $\operatorname{diag}(-1, 1, \ldots, 1) \in (\Gamma \cap s_0^{-1}\Gamma s_0)$ and hence if $B \notin \Gamma_0$, we may take $A = \operatorname{diag}(-1, 1, \ldots, 1)B$. This completes the proof. □

A refinement of this leads to

Proposition 3.2. *$H(GL_n(\mathbb{Z}), GL_n(\mathbb{Q}))$ is generated, as a ring, by $H(SL_n(\mathbb{Z}), M_n^+(\mathbb{Z}))$ together with the elements $(p^{-1}I_n)$ for all primes p. (I_n denotes the $n \times n$ identity matrix.)*

Draft Remark 2. something similar is lemma 2.15 of [2].

Setting

$$H^n = H(GL_n(\mathbb{Z}), GL_n(\mathbb{Q}))$$
$$\underline{H}^n = H(SL_n(\mathbb{Z}), M_n^+(\mathbb{Z})),$$

we therefore view $\underline{H}^n$ as the integral subring of H^n.

While we have defined Hecke algebras $H = H(\Gamma, S)$ only as $\mathbb{Z}$-modules, obviously we can choose coefficients from any ring. In particular, the associated rational algebra is $H_{\mathbb{Q}} = \mathbb{Q} \otimes H$. In an abuse of notation we may simply denote this H as well, with the context making clear that scalars are in $\mathbb{Q}$.

Draft Remark 3. ******** we need to watch to see if we ever do this **********

3.2. **Local Hecke Algebras.** Henceforth, we shall be concerned with the p-part of the Hecke algebra H^n. In the notation of [2], this is defined by

$$H_p^n = H(GL_n(\mathbb{Z}), GL_n(\mathbb{Z}[p^{-1}])) \subset H^n.$$

The subrings H_p^n for all primes p generate H^n.

The integral subring $\underline{H}_p^n$ of H_p^n is

$$\begin{aligned}\underline{H}_p^n &= \langle (\xi) \mid \xi \in M_n(\mathbb{Z}), \det(\xi) = \pm p^\lambda \rangle \\ &= \langle (\operatorname{diag}(p^{i_1}, \dots, p^{i_n})) \mid 0 \le i_1 \le \cdots \le i_n \rangle\end{aligned}$$

where the angle brackets enclose generators as a $\mathbb{Z}$-module. As a ring, H_p^n is generated by $\underline{H}_p^n$ together with the single element $(p^{-1} I_n)$.

Thus the study of the the global Hecke ring H^n reduces to study of the local Hecke rings H_p^n, which in turn can be understood through their integral subrings $\underline{H}_p^n$. However, the remainder of this paper will focus on the p-adic Hecke ring

$$\mathcal{H}_p^n = H(GL_n(\mathbb{Z}_p), GL_n(\mathbb{Q}_p)),$$

and its integral subring

$$\underline{\mathcal{H}}_p^n = \langle (\operatorname{diag}(p^{i_1}, \dots, p^{i_n})) \mid 0 \le i_1 \le \cdots \le i_n \rangle,$$

which together with the single element $(p^{-1} I_n)$ generates $\mathcal{H}_p^n$. This is sufficient for studying the above algebras, though, since $\mathcal{H}_p^n$ (resp. $\underline{\mathcal{H}}_p^n$) is isomorphic to H_p^n (resp. $\underline{H}_p^n$).

To see this isomorphism, let $\Gamma = GL_n(\mathbb{Z})$, $\Gamma_p = GL_n(\mathbb{Z}_p)$, $S = GL_n(\mathbb{Z}[p^{-1}])$, and $S_p = GL_n(\mathbb{Q}_p)$. Then

$$H_p^n = H(\Gamma, S) = \langle \Gamma \operatorname{diag}(p^{i_1}, p^{i_2}, \dots, p^{i_n}) \Gamma \mid i_1 \le i_2 \le \cdots \le i_n \rangle,$$

while

$$\mathcal{H}_p^n = H(\Gamma_p, S_p) = \langle \Gamma_p \operatorname{diag}(p^{i_1}, \dots, p^{i_n}) \Gamma_p \mid i_1 \le i_2 \le \cdots \le i_n \rangle.$$

Now consider the obvious map $H_p^n \to \mathcal{H}_p^n$ induced by taking $\Gamma\xi\Gamma \mapsto \Gamma_p\xi\Gamma_p$ for $\xi = \mathrm{diag}(p^{i_1}, \ldots, p^{i_n})$. It is immediate that this is a vector space isomorphism, so we need only show it preserves multiplication. Since multiplication in the Hecke algebras is defined in terms of multiplication of right coset representatives, it suffices to establish the following:

Lemma 3.3. *If $\alpha \in GL_n(\mathbb{Z}[p^{-1}])$ and $\Gamma\alpha\Gamma = \cup\Gamma\alpha_i$ (disjoint union), then*

$$\Gamma_p\alpha\Gamma_p = \cup\Gamma_p\alpha_i \quad \textit{(disjoint union).} \tag{3.1}$$

Proof. We may assume α has entries in $\mathbb{Z}$, since the general case then follows by multiplication of α by p^k for some $k \in \mathbb{Z}$. Note that for such an α each coset representative α_i has coefficients in $\mathbb{Z}$ and has determinant $\pm p^\lambda$. Suppose that $\Gamma_p\alpha_i \cap \Gamma_p\alpha_j$ is non-empty. Then $\alpha_i\alpha_j^{-1} \in \Gamma_p$, has entries in $\mathbb{Z}[1/p] \cap \mathbb{Z}_p$, and has determinant ± 1, thus $\alpha_i\alpha_j^{-1} \in \Gamma$. Thus $\Gamma\alpha_i = \Gamma\alpha_j$, and so $\alpha_i = \alpha_j$. Therefore the union in (3.1) is disjoint.

To establish the equality in (3.1), note first that $\Gamma_p\alpha_i \subseteq \Gamma_p\alpha\Gamma_p$ is clear. Thus we need only show that if $\beta \in \Gamma_p\alpha\Gamma_p$, then $\beta \in \Gamma_p\alpha_i$ for some i. Without loss of generality, we may assume α is diagonal, since it may be brought into that form by multiplication on the left and right by elements of Γ.

Let $\beta = \gamma_1\alpha\gamma_2$ with $\gamma_1, \gamma_2 \in \Gamma_p$. We may assume $\det(\gamma_2) = 1$, since if otherwise, we can set $\epsilon = \mathrm{diag}(\det(\gamma_2), 1, 1, \ldots, 1) \in \Gamma_p$ and write

$$\gamma_1\alpha\gamma_2 = \gamma_1\alpha\epsilon\epsilon^{-1}\gamma_2 = (\gamma_1\epsilon)\alpha(\epsilon^{-1}\gamma_2),$$

where we've used the fact that diagonal matrices commute. The expressions in parentheses then can be taken as the new γ_1 and γ_2.

Choose a positive integer m such that $p^m\alpha^{-1} \in M_n(p\mathbb{Z})$. By a density argument, pick $\gamma_3 \in \Gamma$, $\delta \in M_n(\mathbb{Z}_p)$, with $\gamma_2 = \gamma_3 + p^m\delta$. Then

$$\beta = \gamma_1\alpha\gamma_2 = \gamma_1\alpha(\gamma_3 + p^m\delta) = \gamma_1(I_n + p^m\alpha\delta\gamma_3^{-1}\alpha^{-1})\alpha\gamma_3.$$

By the choice of m, the last expression in parentheses is in Γ_p. Thus $\beta \in \Gamma_p\alpha\gamma_3$. Since $\Gamma\alpha\gamma_3 = \Gamma\alpha_i$ for some i, this means $\beta \in \Gamma_p\alpha_i$. □

Draft Remark 4. Here we give the required density argument. Let $\Gamma_p^{\pm 1}$ denote the subgroup of Γ_p consisting of matrices with determinant ± 1. We show that Γ is dense in $\Gamma_p^{\pm 1}$.
Let $\gamma_1 \in \Gamma_p^{\pm 1}$. Then using elementary row and column operations, we can express γ_1 as a product of permutation matrices, matrices E which look like the identity but with one off-diagonal entry in $\mathbb{Z}_p$, and a diagonal matrix with determinant ± 1.
For any positive integer m we need to find a matrix $\gamma_2 \in \Gamma$ with $\gamma_1 \equiv \gamma_2 \mod p^m$. To do this, it is sufficient to do so for each matrix mentioned above in the product expressing γ_1. Permutation matrices are already in Γ, and handling the E matrices is easy. The diagonal matrix is a little more involved.
If $D = \operatorname{diag}(d_1, d_2, \dots d_n)$ where $d_i \in \mathbb{Z}_p^\times$, we first find a 2×2 matrix in $GL_2(\mathbb{Z})$ congruent to $\begin{pmatrix} d_1 & 0 \\ 0 & d_1^{-1} \end{pmatrix} \mod p^m$. We do this by picking $a \in \mathbb{Z}$, $a \equiv d_1 \mod p^m$, and then picking $c, d \in \mathbb{Z}$ so that $ad - p^{2m}c = 1$. Then $\begin{pmatrix} a & p^m \\ p^m c & d \end{pmatrix} \in GL_2(\mathbb{Z})$. Let D_1 be the $n \times n$ matrix with this 2×2 block in the upper left and 1's on the diagonal. Then, $D_1^{-1} D \equiv \operatorname{diag}(1, d_1^{-1} d_2, d_3, \dots d_n) \mod p^m$. Continuing 'in this way', we ultimately get $D \equiv D_1 D_2 \dots D_n \mod p^n$ where $D_i \in \Gamma$.

3.3. **Generating Series.** For motivational purposes we recall a generating series for certain elements of the local Hecke ring. As in the previous section, let H_p^n denote the p-part of the Hecke algebra H^n.

Let $d_k^n(p)$ be the diagonal matrix $\operatorname{diag}(\underbrace{1, \dots, 1}_{n-k}, \underbrace{p, \dots, p}_{k})$ and denote by $T_k^n(p)$ the double coset $\Gamma d_k^n(p) \Gamma$ in H_p^n. In the notation of [2] this is $\pi_k^n(p) = (d_k^n(p))$.

Let

$$T(p^\ell) = \sum_{\substack{\alpha \in \Gamma \backslash M_n(\mathbb{Z}) / \Gamma \\ \det \alpha = \pm p^\ell}} \Gamma \, \alpha \, \Gamma \in H_p^n$$

The following is Theorem 3.21 of [7] and Proposition III.2.22 of [2]:

thm:globalhecke **Theorem 3.4.** *The formal Hecke series $\sum_{\ell \geq 0} T(p^\ell) u^\ell$ is a rational function given by:*

$$\sum_{\ell \geq 0} T(p^\ell) u^\ell = \left[\sum_{k=0}^{n} (-1)^k p^{\frac{k(k-1)}{2}} T_k^n(p) u^k \right]^{-1}.$$

Further light is shed on this identity by exploring connections of the local Hecke algebra to an algebra of symmetric polynomials. Andrianov and Zhuravlev [2] give a map ω between the Hecke rings H_p^n, $\underline{H}_p^n$ and certain rings of symmetric polynomials in n variables. Known as the spherical map, ω is first defined on right cosets and, after extension by linearity, is then restricted to the Hecke algebra. While its definition is more complicated than we wish to we state here, we do want to describe the action of the spherical map on certain Hecke operators as a means of motivating why one would study Hecke operators through symmetric polynomial rings.

Let $s_k = s_k(x_1, x_2, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$ be the kth elementary symmetric polynomial. Then under the spherical map ω, we have (Lemma 2.21 of [2] (Chapter 3)) that

$$\omega(\pi_k^n(p)) = p^{\frac{-k(k+1)}{2}} s_k(x_1, \ldots, x_n).$$

so that at least formally (and again identifying $\pi_k^n(p)$ with $T_k^n(p)$)

$$\begin{aligned}
\sum_{\ell \ge 0} \omega(T(p^\ell)) u^\ell &= \left[\sum_{k=0}^{n} (-1)^k p^{\frac{k(k-1)}{2}} \omega(\pi_k^n(p)) u^k \right]^{-1} \\
&= \left[\sum_{k=0}^{n} (-1)^k p^{-k} s_k(x_1, \ldots, x_n) u^k \right]^{-1} \qquad (3.2) \\
&= \prod_{i=1}^{n} \left(1 - p^{-1} x_i u\right)^{-1}.
\end{aligned}$$

This shows how the structure of the "Euler factor" of this generating series of Hecke operators is most easily expressed through the spherical map. Using the spherical map, Andrianov also shows that the p-part of the Hecke algebra, $\underline{H}_p^n$, is isomorphic to the full ring of symmetric polynomials in n variables. With such an intimate connection between Hecke operators and symmetric polynomials we are led to a more comprehensive study.

4. Symmetric polynomials

Based upon the nice action of the spherical map on the elements $\pi_k^n(p)$, one might be inclined to investigate the image of double cosets under the spherical map as a means of characterizing "natural" Hecke operators in the ring of symmetric polynomials. In fact the image of most double cosets is not particularly attractive, so we take another approach.

First, we define a family of symmetric polynomials, denoted $t_k^n(p^\ell)$ and which we will call Hecke operators, whose associated generating series are rational functions — in particular kth exterior powers. We shall see that one subcollection of these operators, the $t_1^n(p^\ell)$, are essentially formal analogs of the classical Hecke operators

$T(p^\ell)$. In particular, the generating series for both collections of operators give rise to essentially the same rational functions.

This section deals only with symmetric polynomials without explicit connections to the Hecke algebra. However, later in the paper we shall see that these polynomials (actually via a representation of the associated algebra) act naturally on the Bruhat-Tits building for SL_n, and reduce to the operators which Serre introduced on trees in the case $n = 2$.

The symmetric group S_n acts naturally on polynomials in n variables. Thus for a polynomial P it makes sense to refer to the stabilizer $\mathrm{Stab}(P)$ in S_n of P. For a polynomial P in n variables, define the *symmetrized polynomial* associated to P, $\mathrm{Sym}_n(P)$, by

$$\mathrm{Sym}_n(P) = \sum_{\sigma \in S_n/\,\mathrm{Stab}(P)} \sigma(P)$$

We understand that if P is a constant, that $\mathrm{Sym}_n(P) = P$.

For our later convenience, we observe the following.

Lemma 4.1. *Let M be a monomial, and $F = \mathrm{Sym}_n(M)$. If M' is another monomial occurring in F, then $F = \mathrm{Sym}_n(M')$.*

Draft Remark 5. *Proof.* Let $H = \mathrm{Stab}(M)$ in S_n. Then $F = \sum_{\sigma \in S_n/H} \sigma(M)$. If M' i s another monomial in F, then $M' = \tau M$ for some $\tau \in S_n$. Since $\mathrm{Stab}(M') = \tau\,\mathrm{Stab}(M)\tau^{-1}$, by definition, we have

$$\mathrm{Sym}_n(M') = \sum_{\varphi \in S_n/\tau H \tau^{-1}} \varphi(M') = \sum_{\varphi \in S_n/\tau H \tau^{-1}} \varphi\tau(M)$$

We need only establish that as φ runs through a set of coset representatives of $S_n/\tau H\tau^{-1}$, $\varphi\tau$ runs through a set of coset representatives for S_n/H. We have that $\varphi\tau H = \varphi'\tau H$ if and only if $\tau^{-1}\varphi^{-1}\varphi'\tau \in H$ from which we see the correspondence is $1-1$. As there are the same number of cosets in both cases, we are done. □

Let m and n be positive integers and denote by $P_n(m)$ the set of partitions of m into n pieces such that a given partition satisfies: $m \geq i_1 \geq \cdots \geq i_n \geq 0$ (and $\sum i_k = m$).

Introduce the lexicographic ordering on $P_n(m)$, and let $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in P_n(m)$. For indeterminates $z_1, z_2, \ldots, z_n$ define

$$hp(0,\dots,0) = 1,$$
$$hp(\mathbf{i}) = hp(i_1, i_2, \dots, i_n) = \sum_{\substack{\mathbf{j}\leq\mathbf{i} \\ \mathbf{j}\in P_n(m)}} \mathrm{Sym}_n(z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}), \text{ and}$$
$$h^n(\ell) = hp(\underbrace{\ell, 0, \dots, 0}_{n}) = \sum_{\mathbf{j}\in P_n(\ell)} \mathrm{Sym}_n(z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}) = \sum_{\substack{\sum j_k=\ell \\ j_k\geq 0}} z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}$$

Note that the $hp(\mathbf{i})$ form a basis for the $\mathbb{Q}$-algebra of symmetric polynomials. However, the polynomials $h^n(\ell)$ appear in a generating series with a particularly simple rational expression.

Proposition 4.2. *The generating series associated to the $h^r(\ell)$ satisfies* prop:basiceuler

$$\sum_{\ell\geq 0} h^r(\ell) u^\ell = [(1-uz_1)\cdots(1-uz_r)]^{-1}$$

Proof. This is essentially obvious:

$$\begin{aligned}
[(1-uz_1)\cdots(1-uz_r)]^{-1} &= \left(\sum_{a_1\geq 0}(uz_1)^{a_1}\right)\cdots\left(\sum_{a_r\geq 0}(uz_r)^{a_r}\right) \\
&= \sum_{\ell\geq 0} u^\ell \cdot \Big[\sum_{\substack{\sum a_i=\ell \\ a_i\geq 0}} z_1^{a_1}\cdots z_r^{a_r}\Big]
\end{aligned}$$

It is clear from the definitions above that the coefficient of u^ℓ in the given expression is $h^r(\ell)$. □

Let $x_1, x_2, \dots, x_n$ be indeterminates. We now define our family of Hecke operators as symmetric polynomials in $\mathbb{Q}[x_1, x_2, \dots, x_n]$: For $1 \leq k \leq n$, define

$$t_k^n(p^\ell) = h^{\binom{n}{k}}(\ell)\Big|_{\substack{z_i\mapsto\sigma_i(x_1x_2\cdots x_k) \\ \sigma_i\in S_n/\mathrm{Stab}(x_1x_2\cdots x_k)}}$$

We observe that the index $[S_n : \mathrm{Stab}(x_1x_2\cdots x_k)]$ is the size of the orbit of $x_1x_2\cdots x_k$ under the group action, that is, the number of distinct monomials in k different variables which one can form with n variables. This is clearly $\binom{n}{k}$, the binomial coefficient. Thus there is one element $\sigma_i \in S_n/\mathrm{Stab}(x_1x_2\cdots x_k)$ to correspond to each of the variables $z_1, \dots, z_{\binom{n}{k}}$.

The above definition is sufficiently complex to warrant an example, however we note before giving detail that $t_k^n(p)$ is nothing more than the k^{th} elementary symmetric polynomial in the variables $x_1, x_2, \dots, x_n$.

Example 4.3. We begin by computing

$$h^r(1) = hp(\underbrace{1, 0, \dots, 0}_{r}) = \mathrm{Sym}_r(z_1) = z_1 + z_2 + \cdots + z_r.$$

Then for example,

$$t_1^n(p) = h^n(1)\Big|_{\substack{z_i \mapsto \sigma_i(x_1) \\ \sigma_i \in S_n/\operatorname{Stab}(x_1)}} = x_1 + x_2 + \cdots + x_n = s_1(x_1, x_2, \dots, x_n),$$

the first elementary symmetric polynomial, and in general

$$t_k^n(p) = h^{\binom{n}{k}}(1)\Big|_{\substack{z_i \mapsto \sigma_i(x_1 x_2 \cdots x_k) \\ \sigma_i \in S_n/\operatorname{Stab}(x_1 x_2 \cdots x_k)}} = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k} = s_k(x_1, x_2, \dots, x_n),$$

the k^{th} elementary symmetric polynomial.

Remark 4.4. Notice that the polynomials $t_k^n(p^\ell)$ are symmetric polynomials in the variables $x_1, x_2, \dots, x_n$. As such,

$$t_k^n(p^\ell) \in \mathbb{Q}[x_1, \dots, x_n]^{\text{sym}} = \mathbb{Q}[s_1, s_2, \dots, s_n].$$

Since $t_k^n(p) = s_k$, we have that the algebra generated by these operators is actually equal to the full ring of symmetric polynomials:

$$\langle t_k^n(p^\ell) \mid \ell \ge 0, k = 1, 2, \dots, n\rangle = \langle t_k^n(p) \mid k = 1, 2, \dots, n\rangle = \mathbb{Q}[s_1, s_2, \dots, s_n]$$

Turning to generating series, we find

prop:generaleuler **Proposition 4.5.** *For $k = 1, 2, \dots, n$, the generating series for the Hecke operators $t_k^n(p^\ell)$ is a rational function:*

$$\sum_{\ell \ge 0} t_k^n(p^\ell) u^\ell = \Big[\prod_{\sigma \in S_n/\operatorname{Stab}(x_1 x_2 \cdots x_k)} (1 - u\sigma(x_1 x_2 \cdots x_k))\Big]^{-1}$$

Remark 4.6. The Euler factors which appear here also appeared in work of Andrianov [1]. He defined a zeta function which encapsulated the algebra structure of the full local Hecke algebra and whose rational expression had denominator equal to the product of all of the denominators above, i.e.,

$$\prod_{k=1}^{n} \prod_{\sigma \in S_n/\operatorname{Stab}(x_1 x_2 \cdots x_k)} (1 - u\sigma(x_1 x_2 \cdots x_k)).$$

However, the numerator of that rational expression was non-trivial and not explicitly given. In this paper, we are producing a family of operators whose generating series are rational functions whose denominators are the individual factors of this product, with numerators 1.

Proof. From Proposition 4.2, we know that

$$\sum_{\ell\geq 0} h^{\binom{n}{k}}(\ell)u^\ell = \Big[\prod_{i=1}^{\binom{n}{k}}(1-uz_i)\Big]^{-1}.$$

But

$$t_k^n(p^\ell) = h^{\binom{n}{k}}(\ell)\Big|_{\substack{z_i\mapsto\sigma_i(x_1x_2\cdots x_k)\\ \sigma_i\in S_n/\operatorname{Stab}(x_1x_2\cdots x_k)}}$$

which completes the proof. □

Example 4.7. Consider the case of $n=4$. We have the four Hecke series:

$$\sum_{\ell\geq 0} t_1^4(p^\ell)u^\ell = [(1-ux_1)(1-ux_2)(1-ux_3)(1-ux_4)]^{-1}$$
$$\sum_{\ell\geq 0} t_2^4(p^\ell)u^\ell = [(1-ux_1x_2)(1-ux_1x_3)(1-ux_1x_4)(1-ux_2x_3)(1-ux_2x_4)(1-ux_3x_4)]^{-1}$$
$$\sum_{\ell\geq 0} t_3^4(p^\ell)u^\ell = [(1-ux_1x_2x_3)(1-ux_1x_2x_4)(1-ux_1x_3x_4)(1-ux_2x_3x_4)]^{-1}$$
$$\sum_{\ell\geq 0} t_4^4(p^\ell)u^\ell = [(1-ux_1x_2x_3x_4)]^{-1}$$

Remark 4.8. Later in this paper we establish a connection between these t_k^n and classical Hecke operators. Accepting such a connection for now, note that the Euler factor corresponding to t_k^n has degree $\binom{n}{k}$ and the monomials are the kth exterior powers of the x_is. Then, observing that the Euler factor corresponding to $\sum t_1^n(p^\ell)u^\ell$ seems most basic to GL_n and that the Euler factor corresponding to $\sum t_k^n(p^\ell)u^\ell$ has degree $\binom{n}{k}$ suggests a correspondence between forms on GL_n and forms on $GL_{\binom{n}{k}}$.

We now illustrate parallels between the polynomial Hecke operators, $t_1^n(p^\ell)$, and those defined through double cosets, $T(p^\ell)$. We first note the following

Lemma 4.9. *Let $s_j = s_j(x_1, x_2, \dots, x_n)$. Then*

$$(1-ux_1)\cdots(1-ux_n) = 1 - s_1u + s_2u^2 + \cdots + (-1)^n s_nu^n.$$

Draft Remark 6. *Proof.*

$$\begin{aligned}(1-ux_1)\cdots(1-ux_n) &= (-1)^n(x_1\cdots x_n)(u-x_1^{-1})\cdots(u-x_n^{-1})\\ &= (-1)^n(x_1\cdots x_n)[u^n - s_1(x_1^{-1},\dots,x_n^{-1})u^{n-1} + s_2(x_1^{-1},\dots,x_n^{-1})u^{n-2}\\ &\qquad + \cdots + (-1)^n s_n(x_1^{-1},\dots,x_n^{-1})u^0]\\ &= (x_1\cdots x_n)[s_n(x_1^{-1},\dots,x_n^{-1}) - us_{n-1}(x_1^{-1},\dots,x_n^{-1}) + \cdots + (-1)^n u^n]\end{aligned}$$

Finally, observe that since $\binom{n}{j} = \binom{n}{n-j}$ is the number of monomials in $s_j(x_1^{-1},\dots,x_n^{-1})$, we have that $(x_1\cdots x_n)s_j(x_1^{-1},\dots,x_n^{-1}) = s_{n-j}(x_1,\dots,x_n)$ for $j\ge 1$ which completes the proof. $\square$

From Proposition 4.2 and the lemma above, we have that

$$\begin{aligned}\sum_{\ell\ge 0} t_1^n(p^\ell)u^\ell &= \Big[\prod_{i=1}^n(1-ux_i)\Big]^{-1}\\ &= \left[\sum_{k=0}^n(-1)^k s_k(x_1,\dots,x_n)u^k\right]^{-1},\end{aligned}$$

which is similar to (3.2) above. In fact, if we apply the algebra automorphism of $\mathbb{Q}[s_1,\dots,s_n]$ induced by taking $s_k(x_1,x_2,\dots,x_n)$ to $p^{-k}s_k(x_1,x_2,\dots,x_n)$, the resulting "Euler" factor would coincide with the earlier one.

prop:sympolys **Proposition 4.10.** *The only extensions to $\mathbb{Q}[x_1,x_2,\dots x_n]$ of the automorphism of $\mathbb{Q}[s_1,s_2\dots,s_n]$ which maps $s_j\mapsto p^{-j}s_j$ are those which map $x_j\mapsto p^{-1}x_{\sigma(j)}$ for some σ in the symmetric group S_n.*

Remark 4.11. 1. The proposition says that the automorphism of $\mathbb{Q}[s_1,s_2,\dots,s_n]$ can be thought of as the restriction (to the subalgebra of symmetric polynomials) of the automorphism induced by $x_i\mapsto p^{-1}x_{\sigma(i)}$ for any $\sigma\in S_n$.

2. We could have hidden this automorphism by defining our Hecke operator $t_k^n(p^\ell)$ as

$$t_k^n(p^\ell) = h^{\binom{n}{k}}(\ell)\Big|_{\substack{z_i\mapsto p^{-k}\sigma_i(x_1x_2\cdots x_k)\\ \sigma_i\in S_n/\operatorname{Stab}(x_1x_2\cdots x_k)}},$$

but the introduction of the power of p into the expression $z_i\mapsto p^{-k}\sigma_i(x_1x_2\cdots x_k)$ would have hardly been motivated at that point. Moreover, as we shall see in the next section, we want to define a representation of this algebra on an apartment associated to $SL_n(\mathbb{Q}_p)$ and then on the building. Since the structure of the apartments are independent of p, the dependence upon p in the operators should appear only at the level of the building.

Proof. (of proposition) Let $\varphi : \mathbb{Q}[x_1, x_2, \dots, x_n] \to \mathbb{Q}[x_1, \dots, x_n]$ be a $\mathbb{Q}$-algebra isomorphism such that $\varphi(s_j(x_1, \dots, x_n)) = p^{-j} s_j(x_1, x_2, \dots, x_n)$. We want to show that $\varphi(x_j) = p^{-1} x_{\sigma(j)}$ for some permutation σ in S_n. Let $\xi_j = \varphi(x_j)$. Then

$$s_j(\xi_1, \dots, \xi_n) = \varphi(s_j(x_1, x_2, \dots, x_n)) = p^{-j} s_j(x_1, x_2, \dots, x_n) = s_j(\frac{x_1}{p}, \dots, \frac{x_n}{p}).$$

Since the polynomials

$$\begin{aligned}\prod_{k=1}^{n}(u - \xi_k) &= \sum_{k=0}^{n}(-1)^k s_k(\xi_1, \dots, \xi_n) u^{n-k} \\ &= \sum_{k=0}^{n}(-1)^k s_k(\frac{x_1}{p}, \dots, \frac{x_n}{p}) u^{n-k} = \prod_{k=1}^{n}(u - \frac{x_k}{p})\end{aligned}$$

have the same roots, the proof is complete. $\square$

5. Action on Buildings and Apartments

There are at least two reasons we defined the Hecke operators in the ring of polynomials. The first is that we produce a natural way of characterizing Andrianov's spherical map — which is instrumental in showing that the local Hecke algebra is isomorphic to the ring of symmetric polynomials. The second is that there is a natural action of this polynomial Hecke algebra on the apartments of the Bruhat-Tits building for SL_n over a local field. It was actually the latter reason which was provided the initial impetus for this investigation. In the first section we give an alternate characterization of the spherical map.

5.1. **A natural characterization of the spherical map.** As in the previous section, $x_1, x_2, \dots, x_n$ are indeterminates and $s_k = s_k(x_1, x_2, \dots, x_n)$ is the kth elementary symmetric polynomial in the x_i. We let $\omega : \underline{H}_p^n \to \mathbb{Q}[s_1, \dots, s_n]$ denote Andrianov's spherical map([2]).

We have previously established a canonical isomorphism between the (integral subrings of the) p-part of the global Hecke algebra $\underline{H}_p^n$ and the local Hecke algebra $(\underline{\mathcal{H}}_p^n)$.

Recalling that $\Gamma_p = GL_n(\mathbb{Z}_p)$ and $d_k^n(p)$ is the diagonal matrix $\mathrm{diag}(\underbrace{1, \dots, 1}_{n-k}, \underbrace{p, \dots, p}_{k})$, we may justifiably denote by $T_k^n(p)$ the double coset $\Gamma_p d_k^n(p) \Gamma_p$ in $\mathcal{H}_p^n$.

Define maps φ_A, φ_B, ψ_A, ψ_B corresponding the the commutative diagrams which follow and which are induced by:

$$
\begin{aligned}
&\varphi_A : \mathcal{H}_p^n \to \mathbb{Q}[s_1, \dots, s_n] : \quad T_k^n(p) \mapsto p^{-k(k-1)/2} t_k^n(p) = p^{-k(k-1)/2} s_k \\
&\varphi_B : \mathcal{H}_p^n \to \mathbb{Q}[s_1, \dots, s_n] : \quad T_k^n(p) \mapsto t_k^n(p) = s_k \\
&\psi_A : \mathbb{Q}[s_1, \dots, s_n] \to \mathbb{Q}[s_1, \dots, s_n] : \quad t_k^n(p) \mapsto p^{-k} t_k^n(p) = p^{-k} s_k \\
&\psi_B : \mathbb{Q}[s_1, \dots, s_n] \to \mathbb{Q}[s_1, \dots, s_n] : \quad t_k^n(p) \mapsto \widehat{t}_k^n(p) = p^{-k(k+1)/2} t_k^n(p) = p^{-k(k+1)/2} s_k
\end{aligned}
$$

We note that $\psi_A \circ \varphi_A = \psi_B \circ \varphi_B$, and that the four maps are ring isomorphisms, mapping well-known (algebraically independent) generators to (algebraically independent) generators. Each composition (together with the canonical isomorphism of $\underline{\mathcal{H}_p^n}$ and $\underline{H_p^n}$) gives an alternate and quite natural interpretation of the spherical map without reference to right cosets. To reveal the significance of these maps, we characterize their action on various Hecke operators.

We note that the diagrams which follow contain information about the action of the Hecke algebra on the Bruhat-Tits building which will be developed fully in the next section. For now, we make only the following remarks. Let Δ_n be the Bruhat-Tits building for $SL_n(\mathbb{Q}_p)$, and let $\mathcal{B}$ (resp. $\mathcal{A}$) denote $\mathbb{Q}$-vector space with basis the vertices of Δ_n (resp. by the vertices of a fixed apartment in Δ_n). The map ρ is a representation of the polynomial Hecke algebra in the building (apartment), defined by the diagrams below and

$$
\rho(t_k^n(p)) = \widetilde{t}_k^n(p) = \begin{cases} t_k^n(p) & \text{if } k < n \\ 1 & \text{if } k = n \end{cases}.
$$

$$
\begin{tikzcd}
 & \underline{\mathcal{H}_p^n} \arrow[dl, "\varphi_A"'] \arrow[dr, "\varphi_B"] \arrow[dd, leftrightarrow, "\cong"] & \\
\mathbb{Q}[s_1, \dots, s_n] \arrow[dd, "\rho"'] \arrow[ddr, "\psi_A"] & & \mathbb{Q}[s_1, \dots, s_n] \arrow[dd, "\rho"] \arrow[ddl, "\psi_B"'] \\
 & \underline{H_p^n} \arrow[dd, "\omega"] & \\
End(\mathcal{A}) & & End(\mathcal{B}) \\
 & \mathbb{Q}[s_1, \dots, s_n] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & T_k^n(p) \arrow[dl, "\varphi_A"'] \arrow[dr, "\varphi_B"] \arrow[dd, leftrightarrow, "\cong"] & \\
p^{-k(k-1)/2} t_k^n(p) \arrow[dd, "\rho"'] \arrow[dddr, "\psi_A"] & & t_k^n(p) \arrow[dd, "\rho"] \arrow[dddl, "\psi_B"'] \\
 & \pi_k^n(p) \arrow[dd, "\omega"] & \\
p^{-k(k-1)/2}\widetilde{t}_k^n(p) & & \widetilde{t}_k^n(p) \\
 & \omega(\pi_k^n(p)) = p^{-k(k+1)/2} t_k^n(p) &
\end{tikzcd}
$$

We want to show how the maps defined above act on our various Hecke operators: $T(p^\ell)$, $t(p^\ell)$, $t_1^n(p^\ell)$. The summary is contained in the commutative diagram below, but we take our time to explore these relations.

$$
\begin{tikzcd}
 & \displaystyle\sum_{\ell\geq 0} T(p^\ell)u^\ell = \left[\sum_{k=0}^{n}(-1)^k p^{\frac{k(k-1)}{2}} T_k^n(p)u^k\right]^{-1} \arrow[dl, "\varphi_A"'] \arrow[dr, "\varphi_B"] \arrow[dd, leftrightarrow, "\cong"] & \\
\displaystyle\sum_{\ell\geq 0} t_1^n(p^\ell)u^\ell = \left[\sum_{k=0}^{n}(-1)^k s_k u^k\right]^{-1} \arrow[dd, "\rho"'] \arrow[dddr, "\psi_A"] & & \displaystyle\sum_{\ell\geq 0} \widehat{t}_1^n(p^\ell)u^\ell = \left[\sum_{k=0}^{n}(-1)^k p^{k(k-1)/2} s_k u^k\right]^{-1} \arrow[dd, "\rho"] \arrow[dddl, "\psi_B"'] \\
 & \displaystyle\sum_{\ell\geq 0} t(p^\ell)u^\ell = \left[\sum_{k=0}^{n}(-1)^k p^{\frac{k(k-1)}{2}} \pi_k^n(p)u^k\right]^{-1} \arrow[dd, "\omega"] & \\
\displaystyle\sum_{\ell\geq 0} T_A(p^\ell)u^\ell = \left[\sum_{k=0}^{n}(-1)^k \widetilde{t}_k^n(p)u^k\right]^{-1} & & \displaystyle\sum_{\ell\geq 0} T_B(p^\ell)u^\ell = \left[\sum_{k=0}^{n}(-1)^k p^{k(k-1)/2}\widetilde{t}_k^n(p)u^k\right]^{-1} \\
 & \displaystyle\sum_{\ell\geq 0} \omega(t(p^\ell))u^\ell = \left[\sum_{k=0}^{n}(-1)^k p^{-k} s_k u^k\right]^{-1} &
\end{tikzcd}
$$

Consider the rational functions associated to the two Hecke series:

$$\sum_{\ell\geq 0} T(p^\ell)u^\ell = \left[\sum_{k=0}^{n}(-1)^k p^{\frac{k(k-1)}{2}} T_k^n(p)u^k\right]^{-1} \quad \text{and} \quad \sum_{\ell\geq 0} t_1^n(p^\ell)u^\ell = \left[\sum_{k=0}^{n}(-1)^k s_k u^k\right]^{-1}.$$

By comparing the rational functions which represent the generating functions, we see that it is quite natural to map (inducing φ_A) $p^{\frac{k(k-1)}{2}} T_k^n(p)$ to $s_k = t_k^n(p)$, i.e., $T_k^n(p)$ to $p^{\frac{-k(k-1)}{2}} t_k^n(p)$.

It is equally natural to simply map (inducing φ_B) $T_k^n(p)$ to $t_k^n(p)$. Here the new operators $\widehat{t_1^n}(p^\ell)$ are defined by the rational function, the image of which under the representation ρ will be a natural Hecke operator on the Bruhat-Tits building for SL_n.

The maps ψ_A and ψ_B are simply elementary automorphisms of the symmetric polynomial ring: ψ_A taking $s_k \mapsto p^{-k}s_k$ (discussed in Proposition 4.10), and ψ_B taking $s_k \mapsto p^{-k(k+1)/2}s_k$.

Since the spherical map ω takes $\pi_k^n(p)$ to $p^{-k(k+1)/2}s_k = p^{-k(k+1)/2}t_k^n(p)$, we see that the composition of the φs with the ψs gives the spherical map in a completely natural way.

5.2. **Buildings, Apartments and Symmetric Polynomials.** In this section we finally associate Hecke operators characterized by symmetric polynomials with operators acting on the building.

In an earlier section, we saw how to label the vertices of an apartment in the Bruhat-Tits building for $SL_n(\mathbb{Q}_p)$ in such a way as to admit a natural action of $\mathbb{Q}[x_1, \dots, x_n]^{\text{sym}}$ on the vertices. Recall that under that action, the monomials act in the following manner: For nonnegative integers a_i and indeterminates x_i, we say that a vertex w is a $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$-*translate* of v if there exists lattices $L_1 \in v$ and $L_2 \in w$ with $L_2 \subset L_1$ and $\{L_1 : L_2\} = \{p^{a_1}, p^{a_2} \dots, p^{a_n}\}$, where the latter notation represents the elementary divisors of L_2 relative to L_1. Note that since vertices are defined by classes of lattices, any vertex v is an $x_1 \cdots x_n$ - translate of itself, i.e. the $x_1x_2\cdots x_n$ - translate of $v = [L]$ is $[pL] = v$.

Fix a vertex in the apartment, labelled v, and let $\mathcal{L} = \oplus\mathbb{Z}_p e_i$ be a lattice whose class determines the vertex v. Then the vertices in the apartment can be viewed in any of the following ways: the homothety class of a lattice $\mathcal{M}$, denoted $[\mathcal{M}]$ with $\{\mathcal{L} : \mathcal{M}\} = \{p^{a_1}, p^{a_2} \dots, p^{a_n}\}$; the class written explicitly in terms of the basis of $\mathcal{L}$: $[\mathbb{Z}_p p^{a_1} e_1 \oplus \cdots \oplus \mathbb{Z}_p p^{a_n} e_n]$; the $x_1^{a_1}\cdots x_n^{a_n}$-translate of v.

Referring to the commutative diagrams of the preceding section, we defined a representation of the polynomial ring (i.e the local Hecke algebra) into the Bruhat-Tits building by

$$\rho(t_k^n(p)) = \widetilde{t}_k^n(p) = \begin{cases} t_k^n(p) & \text{if } k < n \\ 1 & \text{if } k = n \end{cases}.$$

Notice that this is just the automorphism of $\mathbb{Q}[s_1, \ldots, s_n]$ which takes $s_k \mapsto s_k$ (if $k < n$) and maps $s_n \mapsto 1$. Thus any operator (i.e. symmetric polynomial) acts on $\mathcal{A}$ by translation (extended linearly) as described above. Thus we can define Hecke operators $T_A(p^\ell)$ acting on $\mathcal{A}$ by taking $T_A(p^\ell) = \rho(t_1^n(p^\ell))$. Then from Proposition 4.5ff, we have that

$$\sum_{\ell \geq 0} T_A(p^\ell) u^\ell = \left[\sum_{k=0}^{n} (-1)^k \widetilde{t}_k^n(p) u^k \right]^{-1}.$$

Moreover, it is clear from the definition that

$$t_1^n(p^\ell) = \sum_{\substack{\sum j_k = \ell \\ j_k \geq 0}} x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}.$$

We see that the action of $\mathrm{Sym}_n(x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n})$ defines an operator $T_A(p^{j_1}, \ldots, p^{j_n})$ which acts on the vertex $v = [\mathcal{L}]$ by

$$T_A(p^{j_1}, \ldots, p^{j_n})([\mathcal{L}]) = \sideset{}{^*}\sum_{\{\mathcal{L}:\mathcal{M}\} = \{p^{j_1}, \ldots, p^{j_n}\}} [\mathcal{M}]$$

where the sum is over all sublattices $\mathcal{M}$ relative to the (unordered) basis $\{e_1, \ldots, e_n\}$ of $\mathcal{L}$. Then

$$T_A(p^\ell) = \rho(t_1^n(p^\ell) = \rho(\varphi_A(T(p^\ell))) = \sum_{\substack{\sum a_k = \ell \\ a_k \geq 0}} T_A(p^{a_1}, \ldots, p^{a_n}).$$

The natural definition of a Hecke operator acting on the Bruhat-Tits building is very similar. With the lattice $\mathcal{L}$ fixed as above, we recall [7] that there is a $1-1$ correspondence between right cosets $\Gamma_p \xi_\nu$ in $\Gamma_p \operatorname{diag}(p^{a_1}, \ldots, p^{a_n}) \Gamma_p$ and lattices $\mathcal{M}$ in $V = \mathcal{L} \otimes \mathbb{Q}_p$ with elementary divisors $\{\mathcal{L} : \mathcal{M}\} = \{p^{a_1}, \ldots, p^{a_n}\}$.

Thus it is natural to define the Hecke operator $T_B(p^{a_1}, \ldots, p^{a_n})$ (acting on $\mathcal{B}$) as

$$T_B(p^{a_1}, \ldots, p^{a_n})([\mathcal{L}]) = \sum_{\{\mathcal{L}:\mathcal{M}\} = \{p^{a_1}, \ldots, p^{a_n}\}} [\mathcal{M}]$$

We also define

$$T_B(p^\ell) = \sum_{\mathbf{i} \in P_n(\ell)} T_B(p^{i_1}, \ldots, p^{i_n})$$

It is important to note that since the ring structure of the Hecke algebra H_p^n is characterized [7] precisely in terms of elementary divisors of lattices on V, we have for free that the generating series for the operators $T_B(p^\ell)$ is expressible as the same rational function as the generating series for the classical Hecke operators $T(p^\ell)$, namely

$$\sum_{\ell \geq 0} T_B(p^\ell) u^\ell = \left[\sum_{k=0}^{n} (-1)^k p^{k(k-1)/2} \widetilde{t}_k^n(p) u^k \right]^{-1}.$$

That is,

$$T_B(p^\ell) = \rho(\varphi_B(T(p^\ell))).$$

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