NEWFORMS OF HALF-INTEGRAL WEIGHT

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Abstract. Using a characterization of half-integral weight newforms which originated with Serre and Stark, we consider the structure of the space of newforms and the multiplicity with which half-integral weight newforms (associated to a given integral weight newform) occur. On the full space (not the Kohnen subspace) of cusp forms of squarefree level, we show that half-integral weight newforms associated to a given integral weight newform occur with multiplicity one or two, and of those occurring with multiplicity two, one newform is associated with the Kohnen subspace. Moreover, we give an explicit decomposition of the space of half-integral weight newforms which demonstrates that newforms of squarefree level $4N$ can correspond to integral weight newforms of any level dividing $2N$.

For nonsquarefree levels, we give a decomposition of Ueda's spaces of newforms in terms of the spaces of newforms defined à la Serre and Stark. In particular, the decomposition shows how newforms defined by Ueda of level $4N$ can correspond to integral weight newforms of any level dividing $2N$. More importantly, this decomposition explicitly connects the notions of half-integral weight newforms as defined by Serre and Stark with those defined by Kohnen and Ueda.

1. Background

Two very different definitions of a newform of half-integral weight are present and continued to be developed in the literature. The first definition originated with Serre and Stark for forms of weight 1/2 [5], and is analogous to the definition of newform for integral weight forms, which uses forms of lower level and shifts of such forms ($f(\tau) \mapsto f \mid B_d = f(d\tau)$) to characterize the notion of oldforms. The second definition originated with Kohnen for half-integral weight forms of squarefree level [1], who used forms of lower level and their image under the U_{m^2} operator ($f = \sum a_n e^{2\pi i n\tau} \mapsto f \mid U_{m^2} = \sum a_{nm^2} e^{2\pi i n\tau}$) to define the notion of oldforms. The choice of the U_{m^2} operator over the shift operator B_d seems a propitious one, since the U operator commutes with the action of the Shimura lift, while the shift operator B does not. More to the point, Kohnen was able to develop a newform theory on a distinguished subspace of the full space of cusp forms (now referred to as the Kohnen subspace), and obtained a multiplicity-one result (with respect to Hecke eigenvalues) for half-integral weight newforms in this subspace. Even nicer, the multiplicity-one result was established by showing that there is a one-to-one correspondence between newforms of level $4N$ in the subspace and the newforms of integral weight of level N.

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Several others provided further generalizations of the newform theory using both definitions of half-integral weight newforms ([1], [3], [4], [6], [7],[12]). By far the most dramatic development of this theory was obtained recently by Ueda [12]. In this paper, he obtains for arbitrary level a multiplicity-one theorem on a certain subspace of the Kohnen subspace. The generalization from squarefree level to arbitrary level is not at all obvious, and requires very different accommodations for primes dividing the level with multiplicity greater than one. One curious phenomenon which Ueda mentions is that while he was successful in obtaining a multiplicity-one result, newforms of level $4N$ sometimes correspond (under the Shimura correspondence) to newforms of level a proper divisor of $2N$. Ueda continues to refine his subspace in an attempt to remedy this issue.

In joint work with Walling [7], in which we take the alternate definition of newform and work in the full space of cusp forms and not the Kohnen subspace, we determine in two special cases precisely how it is possible for newforms of half-integral weight and squarefree level to correspond to integral weight newforms of any level dividing the expected level given by the Shimura lift. We conjectured that those special cases should generalize.

This paper has two main thrusts. The first is to continue the investigation initiated in [7]. While the desire to obtain multiplicity-one results has directed a great deal of research towards a careful development of the notion of newforms within the Kohnen subspace, a very natural question is to explore whether the multiplicities with which newforms occur on the full space can be determined. In this paper we answer this question for squarefree level. The rather pleasing answer (see Theorem 2.3) is that newforms of half-integral weight correspond to integral weight newforms with multiplicity one or two, and those that occur with multiplicity two can be identified with Kohnen subspaces in such a way that if the full space of newforms (of level $4N$) is is decomposed into its Kohnen component and a non-Kohnen component, the newforms in each (corresponding to integral weight newforms of level N or $2N$) occur with multiplicity one.

In the second part of the paper we consider nonsquarefree levels and demonstrate a relation between Ueda's space of newforms and our own (see Theorem 3.1 and Corollary 3.7), by decomposing Ueda's space of newforms into subspaces defined using the alternate characterization of newform. In particular, these results make clear why in certain circumstances, Ueda's newforms often correspond to integral weight newforms whose level is lower than expected. It also brings closer together what were apparently two very different notions of a half-integral weight newform. These results are obtained for both the full space of cusp forms and the Kohnen subspace.

2. Decomposition Theorems — squarefree level

Let N be an odd positive integer and ψ an even Dirichlet character defined modulo $4N$. For an odd integer $k \geq 3$, we denote by $S_{k/2}(4N, \psi)$ the space of cusp forms of weight $k/2$ and character ψ for $\Gamma_0(4N)$. When $k = 3$, we take as usual $S_{3/2}(4N, \psi)$ to be the orthogonal complement in the full space of cusp forms of the space generated by theta series attached to quadratic forms in one variable with spherical harmonic (see [12]). For a positive integer M, an integer $\ell \geq 2$, and a Dirichlet character φ defined modulo M, let $\mathfrak{S}_\ell(M, \varphi)$ denote the space of cusp forms of weight ℓ and character φ for $\Gamma_0(M)$. The rest of the notation and the definition of standard operators like the Hecke operators can be found in [12] or [8].

2.1. **An Internal Decomposition.** We recall (see [6]) that the space generated by the oldforms, denoted $S^-_{k/2}(4N, \psi)$, is defined as

$$S^-_{k/2}(4N, \psi) = \sum_{\substack{q \mid N \\ \mathrm{cond}(\psi) \mid 4N/q}} S_{k/2}(4N/q, \psi) + \sum_{\substack{q \mid N \\ \mathrm{cond}(\psi\left(\frac{q}{*}\right)) \mid 4N/q}} S_{k/2}(4N/q, \psi\left(\tfrac{q}{*}\right)) \mid B_q \tag{2.1}$$

where B_q is the shift operator taking $f(\tau)$ to $(f \mid B_q)(\tau) = f(q\tau)$. The subspace generated by the newforms, denoted $S^+_{k/2}(4N, \psi)$ is the orthogonal complement of $S^-_{k/2}(4N, \psi)$ in the full space $S_{k/2}(4N, \psi)$. Our first theorem provides an internal decomposition of the space $S_{k/2}(4N, \psi)$ into sums of spaces of newforms when the level N is odd and squarefree.

It turns out to be easier to handle first the case where the character ψ is trivial and to generalize to nontrivial characters through another means.

Theorem 2.1. *Let N be an odd squarefree integer, and ψ the trivial character modulo $4N$. Then*

$$S_{k/2}(4N, 1) = S_{k/2}(4N, \psi) = \bigoplus_{d \mid N} S^+_{k/2}(4d).$$

Remark 2.2. It is completely straightforward from the definitions of the spaces of oldforms and newforms to show that $S_{k/2}(4N, 1) = \sum_{d \mid N} S^+_{k/2}(4d)$. Moreover, if these were spaces of integral weight newforms, such a sum would obviously be direct (in fact orthogonal) since newforms of distinct levels have infinitely many distinct eigenvalues under the Hecke operators. Unfortunately for half-integral weight forms, this is very far from true. In fact to describe this phenomenon more clearly is a major focus of this paper. The special cases where N is a prime or the product of two distinct primes were established in [7].

Proof. For notational ease within this proof, we suppress the notation for the weight and character, and write $S(4d)$ for $S_{k/2}(4d, 1)$ and $S^\pm(4d)$ for $S^\pm_{k/2}(4d, 1)$. Write $N = p_1 p_2 \cdots p_r$, the product of distinct odd primes. We always have

$$S(4N) = S^+(4N) \oplus S^-(4N),$$

and since $\psi = 1$,

$$S^-(4N) = \sum_{i=1}^{r} S(4p_1 \cdots p_r/p_i).$$

Thus we need to show that

$$S^-(4N) = \bigoplus_{\substack{d|N \\ d<N}} S^+_{k/2}(4d).$$

As we remarked above, a trivial induction shows that

$$S^-(4N) = \sum_{\substack{d|N \\ d<N}} S^+_{k/2}(4d).$$

We rewrite this sum as

$$\sum_{k=0}^{r-1} \sum_{\substack{0\le\nu_i\le1 \\ \sum\nu_i=k}} S^+(4p_1^{\nu_1}p_2^{\nu_2}\cdots p_r^{\nu_r}) \tag{2.2}$$

We first show that for fixed k, the inner sum is direct. There are two cases.

Case 1. For $k = 0$, there is only one summand, so the result is clear. When $k = 1$, $r \ge 2$ and the sum becomes $\sum_{i=1}^r S^+(4p_i)$. Suppose that $\sum f_i = 0$ with $f_i \in S^+(4p_i)$. Then we may assume there exists an index i so that f_i is not zero. Then

$$f_i = -\sum_{j\ne i} f_j \in S^+(4p_i) \cap S(4p_1\cdots p_r/p_i) \subset S(4) \subset S^-(4p_i),$$

which implies $f_i = 0$, a contradiction.

Case 2. In the remaining case we have $2 \le k < r$. This time we employ the trace operator (adjoint to the inclusion) to achieve our goal. For properties of the trace operator, see [6] or [5].

Suppose that $\sum_{\sum\nu_i=k} S^+(4p_1^{\nu_1}\cdots p_r^{\nu_r})$ is not direct. Then there exists a nontrivial dependence relation of the form $\sum f_{j_1\cdots j_k} = 0$ with $f_{j_1\cdots j_k} \in S^+(4p_{j_1}\cdots p_{j_k})$. Fix an index $i_1\cdots i_k$ for which $f_{i_1\cdots i_k} \ne 0$, and put $\mathrm{Tr} = \mathrm{Tr}^{4N}_{4p_{i_1}\cdots p_{i_k}}$. We shall show that $\sum f_{j_1\cdots j_k} \mid \mathrm{Tr} = f_{i_1\cdots i_k} = 0$, providing our contradiction.

Certainly

$$f_{i_1\cdots i_k} \mid \mathrm{Tr} = f_{i_1\cdots i_k} \mid \mathrm{Tr}^{4N}_{4p_{i_1}\cdots p_{i_k}} = f_{i_1\cdots i_k}$$

since $f_{i_1\cdots i_k}$ has level $4p_{i_1}\cdots p_{i_k}$. Now consider $f_{j_1\cdots j_k} \ne f_{i_1\cdots i_k}$. Let $D = \gcd(p_{i_1}\cdots p_{i_k}, p_{j_1}\cdots p_{j_k})$, and write $p_{i_1}\cdots p_{i_k} = DI$ and $p_{j_1}\cdots p_{j_k} = DJ$. Note that I and J are both greater than 1, since $\{i_1,\ldots,i_k\} \ne \{j_1,\ldots,j_k\}$. Finally, write $N = p_1\cdots p_r = DIJK$ for some integer K.

Then

$$\begin{aligned}
f_{j_1\cdots j_k} \mid \mathrm{Tr}^{4N}_{4p_{i_1}\cdots p_{i_k}} &= f_{j_1\cdots j_k} \mid \mathrm{Tr}^{4DIJK}_{4DI} \\
&= f_{j_1\cdots j_k} \mid \mathrm{Tr}^{4DJK}_{4D} \text{ since } \gcd(I, JK) = 1
\end{aligned}$$

and letting p be any prime dividing J (recall $J > 1$)

$$\begin{aligned}
&= f_{j_1\cdots j_k} \mid \mathrm{Tr}^{4DJK}_{4DJ} \mid \mathrm{Tr}^{4DJ}_{4DJ/p} \mid \mathrm{Tr}^{4DJ/p}_{4D} \\
&= f_{j_1\cdots j_k} \mid \mathrm{Tr}^{4DJ}_{4DJ/p} \mid \mathrm{Tr}^{4DJ/p}_{4D} \text{ since } f_{j_1\cdots j_k} \text{ has level } 4DJ \\
&= 0 \mid \mathrm{Tr}^{4DJ/p}_{4D} = 0 \text{ by Theorem 5.2 of [6].}
\end{aligned}$$

So we now have that

$$S^-(4N) = \sum_{k=0}^{r-1} \bigoplus_{\substack{0\le \nu_i \le 1 \\ \sum \nu_i = k}} S^+(4p_1^{\nu_1}p_2^{\nu_2}\cdots p_r^{\nu_r}) \tag{2.3}$$

We now show that the outer sum is direct. Suppose not. Then there exists a nontrivial dependence relation

$$f_0 + f_1 + \cdots + f_{r-1} = 0 \qquad \text{where} \qquad f_k \in \bigoplus_{\substack{0\le \nu_i \le 1 \\ \sum \nu_i = k}} S^+(4p_1^{\nu_1}p_2^{\nu_2}\cdots p_r^{\nu_r})$$

Let ν^* be the largest index so that $f_{\nu^*} \neq 0$, and write

$$f_{\nu^*} = \sum g_{i_1\cdots i_{\nu^*}} \qquad \text{where} \qquad g_{i_1\cdots i_{\nu^*}} \in S^+(4p_{i_1}\cdots p_{i_{\nu^*}})$$

Choose any index $k_1\cdots k_{\nu^*}$ for which $g_{k_1\cdots k_{\nu^*}} \neq 0$, and consider the trace operator $\mathrm{Tr} = \mathrm{Tr}^{4N}_{4p_{k_1}\cdots p_{k_{\nu^*}}}$. Exactly as in the first part of the proof, we see that

$$f_{\nu^*} \mid \mathrm{Tr} = \sum g_{i_1\cdots i_{\nu^*}} \mid \mathrm{Tr}^{4N}_{4p_{k_1}\cdots p_{k_{\nu^*}}} = g_{k_1\cdots k_{\nu^*}}$$

Now we claim that for $\nu < \nu^*$, $f_\nu \mid \mathrm{Tr} \in S^-(4p_{k_1}\cdots p_{k_{\nu^*}})$. This will imply that

$$f_{\nu^*} \mid \mathrm{Tr} = g_{k_1\cdots k_{\nu^*}} = -(f_1 + \cdots + f_{\nu^*-1}) \mid \mathrm{Tr} \in S^-(4p_{k_1}\cdots p_{k_{\nu^*}}),$$

forcing $g_{k_1\cdots k_{\nu^*}} = 0$, a contradiction.

To establish the claim, we first note that essentially by definition of the trace operator, $f_\nu \mid \mathrm{Tr} \in S(4p_{k_1}\cdots p_{k_{\nu^*}})$. For $\nu < \nu^*$, write

$$f_\nu = \sum g_{\ell_1\cdots \ell_\nu} \qquad \text{where} \qquad g_{\ell_1\cdots\ell_\nu} \in S^+(4p_{\ell_1}\cdots p_{\ell_\nu})$$

Let $D = \gcd(p_{\ell_1}\cdots p_{\ell_\nu}, p_{k_1}\cdots p_{k_{\nu^*}})$, and write $p_{k_1}\cdots p_{k_{\nu^*}} = DK$ and $p_{\ell_1}\cdots p_{\ell_\nu} = DL$ for integers K and L. Note that $K > 1$ since $\nu < \nu^*$, but L could equal 1. Finally write $N = p_1\cdots p_r = DKLM$ for some integer M, and note that $\gcd(K, LM) = 1$.

There are two cases. If $L=1$, then $D=p_{\ell_1}\cdots p_{\ell_\nu}$ is a proper divisor of $p_{k_1}\cdots p_{k_{\nu^*}}$. This implies that $g_{\ell_1\cdots\ell_\nu} \mid \mathrm{Tr} = g_{\ell_1\cdots\ell_\nu}$ since $g_{\ell_1\cdots\ell_\nu}$ has level dividing $p_{k_1}\cdots p_{k_{\nu^*}}$. Moreover, since D is a proper divisor of $p_{k_1}\cdots p_{k_{\nu^*}}$, $g_{\ell_1\cdots\ell_\nu}\in S^-(4p_{k_1}\cdots p_{k_{\nu^*}})$.

If $L>1$, then let p be any prime dividing L. Then

$$\begin{aligned} g_{\ell_1\cdots\ell_\nu} \mid \mathrm{Tr} &= g_{\ell_1\cdots\ell_\nu} \mid \mathrm{Tr}^{4DLKM}_{4DK} \\ &= g_{\ell_1\cdots\ell_\nu} \mid \mathrm{Tr}^{4DLM}_{4D} \text{ since } \gcd(K,LM)=1 \\ &= g_{\ell_1\cdots\ell_\nu} \mid \mathrm{Tr}^{4DLM}_{4DL} \mid \mathrm{Tr}^{4DL}_{4DL/p} \mid \mathrm{Tr}^{4DL/p}_{4D} \\ &= g_{\ell_1\cdots\ell_\nu} \mid \mathrm{Tr}^{4DL}_{4DL/p} \mid \mathrm{Tr}^{4DL/p}_{4D} \text{ since the level of } g_{\ell_1\cdots\ell_\nu} \text{ is } 4DL \\ &= 0 \mid \mathrm{Tr}^{4DL/p}_{4D} = 0 \text{ by Theorem 5.2 of [6].} \end{aligned}$$

Thus $f_\nu \mid \mathrm{Tr}\in S^-(4p_{k_1}\cdots p_{k_{\nu^*}})$ which completes the proof of the claim, and of the theorem. □

2.2. **An External Decomposition.** Next we give a correspondence between newforms of half-integral weight and newforms of integral weight. For squarefree level, we shall see that half-integral weight newforms occur with multiplicity 1 or 2, and those occurring with multiplicity 2 are precisely those with at least one member in the Kohnen subspace of newforms.

Let N be an odd positive squarefree integer, and $k\ge 3$ and odd integer. Ueda [10, §3] defines an abstract Hecke algebra $H(N)$ and two representations of it, one in the space of integral weight cusp forms, the other in the space of half-integral weight cusp forms. Isomorphisms (as $H(N)$-modules) between spaces of cusp forms of integral and half-integral weight have the property that an eigenform for a half-integral weight Hecke operator $\tilde{T}_{k/2}(n^2)$ (with $(n,2N)=1$) corresponds to an eigenform for the integral weight Hecke operator $T_{k-1}(n)$ which has the same eigenvalue as its half-integral weight counterpart. Since the Hecke eigenvalues for $T_{k-1}(p)$ for almost all primes p characterize integral weight newforms precisely, we obtain our multiplicity results as a consequence.

In particular, we have the

Theorem 2.3. *Let N be an odd positive squarefree integer, and $k\ge 3$ and odd integer. Then*

$$S^+_{k/2}(4N,1)\cong\bigoplus_{d\mid N}\left[\mathfrak{S}^+_{k-1}(2d)\oplus 2\mathfrak{S}^+_{k-1}(d)\right]$$

where the isomorphism is as modules for the Hecke algebra $H(N)$.

Remark 2.4. Note that this theorem says that newforms of half-integral weight, "squarefree level" $4N$ and trivial character correspond to integral weight newforms of all possible levels dividing $2N$, $2N$ being the level to which they correspond under any Shimura lift. It is also remarkable to note that viewed in the opposite direction, the theorem says that given any newform F of squarefree level d and trivial character, there exist half-integral weight *newforms* of level $2Md$ or $4Md$ having the same eigenvalues as F for any squarefree integer M prime to $2d$. Note that while there obviously exist cusp forms of arbitrarily high levels with the same eigenvalues as F, it is surprising that newforms should exist with this behavior. In particular, such newforms cannot arise as character twists of half-integral weight newforms of level $4d$. Moreover, we note that all newforms in $S^+_{k/2}(4N,1)$ which correspond to integral weight newforms of level $2d$ for any $d \mid N$, do in fact occur with multiplicity one.

Before proving the theorem, we state a corollary which connects the notion of newform developed by Kohnen with the one used in this paper which parallels the one for integral weight forms. With the notation as above, denote the Kohnen subspace by $S_{k/2}(4N,1)_K$ and the corresponding subspace of newforms (see [1] for definitions) by $S^{new}_{k/2}(4N,1)_K$. By Theorem 2.4 of [7], we know that $S^{new}_{k/2}(4N,1)_K \subset S^+_{k/2}(4N,1)$. Let $S^{new}_{k/2}(4N,1)_{NK}$ (NK meaning nonKohnen) be the orthogonal complement of $S^{new}_{k/2}(4N,1)_K$ in $S^+_{k/2}(4N,1)$. Then both $S^{new}_{k/2}(4N,1)_K$ and $S^{new}_{k/2}(4N,1)_{NK}$ are Hecke (i.e., $H(N)$) submodules of $S^+_{k/2}(4N,1)$. From Kohnen's work [1], we know that as $H(N)$-modules $S^{new}_{k/2}(4d,1)_K \cong \mathfrak{S}^+_{k-1}(d,1)$ for any divisor d of N. In particular, we obtain the

Corollary 2.5. *With the notation as above,*

$$\begin{aligned} S^+_{k/2}(4N,1) &= S^{new}_{k/2}(4N,1)_K \oplus S^{new}_{k/2}(4N,1)_{NK} \\ &\cong \bigoplus_{d|2N} \mathfrak{S}^+_{k-1}(d) \oplus \bigoplus_{e|N} \mathfrak{S}^+_{k-1}(e) \\ &\cong \bigoplus_{d|2N} \mathfrak{S}^+_{k-1}(d) \oplus \bigoplus_{e|N} S^{new}_{k/2}(4e,1)_K \end{aligned}$$

In particular,

$$S^{new}_{k/2}(4N,1)_{NK} \cong \mathfrak{S}^+_{k-1}(2N) \oplus \mathfrak{S}^+_{k-1}(N) \oplus \bigoplus_{\substack{d|N \\ d<N}} \left[\mathfrak{S}^+_{k-1}(2d) \oplus 2\mathfrak{S}^+_{k-1}(d)\right]$$

Proof. (of Theorem 2.3) Once again for convenience we shall abbreviate $S^{\pm}_{k/2}(4N,1)$ to $S^{\pm}(4N)$, and we shall also abbreviate $\mathfrak{S}^{\pm}_{k-1}(M,1)$ to $\mathfrak{S}^{\pm}(M)$. The basic idea of the proof is quite straightforward. First, by the corollary in §3 of [10], we obtain

$$S_{k/2}(4N,1) = S^+_{k/2}(4N,1) \oplus S^-_{k/2}(4N) \cong \mathfrak{S}_{k-1}(2N,1).$$

Then we use the integral weight newform theory to decompose $\mathfrak{S}_{k-1}(2N,1)$ and induction with Theorem 2.1 above to identify the oldform parts. Finally, using semisimplicity of the Hecke algebra, we cancel factors yielding the result.

Write $N = p_1 \cdots p_r$ where the p_i are distinct odd primes. Let $\sigma_0(n)$ denote the number of positive divisors of n. Then as modules for the Hecke algebra $H(N)$, integral weight newform theory yields

$$\begin{aligned}
\mathfrak{S}(2p_1\cdots p_r) &\cong \bigoplus_{d|2p_1\cdots p_r} \sigma_0\left(\frac{2p_1\cdots p_r}{d}\right)\mathfrak{S}^+(d)\\
&\cong \bigoplus_{d|p_1\cdots p_r}\left[\sigma_0\left(\frac{p_1\cdots p_r}{d}\right)\mathfrak{S}^+(2d)\oplus\sigma_0\left(\frac{2p_1\cdots p_r}{d}\right)\mathfrak{S}^+(d)\right]\\
&\cong \bigoplus_{d|p_1\cdots p_r}\sigma_0\left(\frac{p_1\cdots p_r}{d}\right)\left[\mathfrak{S}^+(2d)\oplus 2\mathfrak{S}^+(d)\right]\\
&= \bigoplus_{k=0}^{r}\bigoplus_{\substack{0\le\nu_i\le 1\\ \sum\nu_i=k}} 2^{r-k}\left[\mathfrak{S}^+(2p_1^{\nu_1}\cdots p_r^{\nu_r})\oplus 2\mathfrak{S}^+(p_1^{\nu_1}\cdots p_r^{\nu_r})\right]
\end{aligned}\tag{2.4}$$

On the other hand, by Theorem 2.1, we have

$$\begin{aligned}
S(4p_1^{\nu_1}\cdots p_r^{\nu_r}) &= \bigoplus_{d|p_1^{\nu_1}\cdots p_r^{\nu_r}} S^+(4d) = \bigoplus_{\ell=0}^{r}\bigoplus_{\substack{0\le\mu_i\le 1\\ \sum\mu_i=\ell}} S^+(4p_1^{\mu_1}\cdots p_r^{\mu_r})\\
&= S^+(4p_1\cdots p_r)\oplus\bigoplus_{\ell=0}^{r-1}\bigoplus_{\substack{0\le\mu_i\le 1\\ \sum\mu_i=\ell}} S^+(4p_1^{\mu_1}\cdots p_r^{\mu_r})
\end{aligned}\tag{2.5}$$

Now we proceed by induction on r. The base case $r = 0$ being trivial, we assume that $r \ge 1$ and

$$S^+(4p_1^{\mu_1}\cdots p_r^{\mu_r}) \cong \bigoplus_{d|p_1^{\mu_1}\cdots p_r^{\mu_r}}\left[\mathfrak{S}^+(2d)\oplus 2\mathfrak{S}^+(d)\right]$$

for any set of primes with $0 \le \mu_i \le 1$ and $\sum \mu_i = \ell < r$. We rewrite the above as

$$S^+(4p_1^{\mu_1}\cdots p_r^{\mu_r}) \cong \bigoplus_{j=0}^{\ell}\bigoplus_{\substack{0\le m_i\le\mu_i\\ \sum m_i=j}}\left[\mathfrak{S}^+(2p_1^{m_1}\cdots p_r^{m_r})\oplus 2\mathfrak{S}^+(p_1^{m_1}\cdots p_r^{m_r})\right].$$

Combining the above with equations (2.4) and (2.5), we obtain

(2.6)

$$S^+(4p_1\cdots p_r)\oplus\bigoplus_{\ell=0}^{r-1}\bigoplus_{\substack{0\le\mu_i\le1\\ \sum\mu_i=\ell}}\bigoplus_{j=0}^{\ell}\bigoplus_{\substack{0\le m_i\le\mu_i\\ \sum m_i=j}}\left[\mathfrak{S}^+(2p_1^{m_1}\cdots p_r^{m_r})\oplus 2\mathfrak{S}^+(p_1^{m_1}\cdots p_r^{m_r})\right]$$
$$\cong\bigoplus_{k=0}^{r}\bigoplus_{\substack{0\le\nu_i\le1\\ \sum\nu_i=k}}2^{r-k}\left[\mathfrak{S}^+(2p_1^{\nu_1}\cdots p_r^{\nu_r})\oplus 2\mathfrak{S}^+(p_1^{\nu_1}\cdots p_r^{\nu_r})\right]$$

Fix a sequence $\nu_1,\dots,\nu_r$ with $0\le\nu_i\le1$ and $\sum\nu_i=k<r$. Note that the summand $[\mathfrak{S}^+(2p_1^{\nu_1}\cdots p_r^{\nu_r})\oplus 2\mathfrak{S}^+(p_1^{\nu_1}\cdots p_r^{\nu_r})]$ occurs with multiplicity 2^{r-k} on the RHS of equation (2.6). We shall show that it occurs with multiplicity $2^{r-k}-1$ on the LHS. Then using semisimplicity of the Hecke algebra, we cancel like terms on both sides of (2.6) to obtain

$$S^+(4p_1\cdots p_r)\cong\bigoplus_{k=0}^{r}\bigoplus_{\substack{0\le\nu_i\le1\\ \sum\nu_i=k}}\left[\mathfrak{S}^+(2p_1^{\nu_1}\cdots p_r^{\nu_r})\oplus 2\mathfrak{S}^+(p_1^{\nu_1}\cdots p_r^{\nu_r})\right]$$
$$=\bigoplus_{d|N}\left[\mathfrak{S}^+(2d)\oplus 2\mathfrak{S}^+(d)\right]$$

as required. We need only establish the claim concerning the multiplicity.

To make an accurate count we need only count the number of times the r-tuple $(m_1,\dots,m_r)=(\nu_1,\dots,\nu_r)$ occurs on the LHS. In the notation of (2.6), we make two simple observations: For $1\le i\le r$, we have $\nu_i=m_i\le\mu_i$ and $j=k=\sum_{i=1}^r\nu_i=\sum_{i=1}^r m_i\le\sum_{i=1}^r\nu_i=\ell$. Also recall that all entries in the r-tuples are zeros or ones. We need to count the number of r-tuples $\boldsymbol{\mu}=(\mu_1,\dots,\mu_r)$ which can give rise to the r-tuple $\mathbf{m}=(m_1,\dots,m_r)$ subject to the above constraints. We count as a function of ℓ.

From the inequality above, we see that $\ell\ge k$. If $\ell=k$, $k=\sum_{i=1}^r m_i=\sum_{i=1}^r\nu_i=\ell$ which says there is only one possible r-tuple $\boldsymbol{\mu}$ which admits $\mathbf{m}$. For $\ell=k+1$, we know that any admissible $\boldsymbol{\mu}$ has ones everywhere that $\mathbf{m}$ does, plus exactly one more. Since $k=\sum_{i=1}^r m_i$, there are $n-k$ possible indices for the extra one, and hence $n-k$ possible r-tuples $\boldsymbol{\mu}$ which admit $\mathbf{m}$. In general, when $\ell=k+s$ $(0\le s\le n-k-1)$, any admissible $\boldsymbol{\mu}$ has s additional ones where $\mathbf{m}$ has zeros, and there are $n-k$ indices from which to choose where to place them. Of course, this means that there are $\binom{n-k}{s}$ r-tuples $\boldsymbol{\mu}$ which admit $\mathbf{m}$. Thus in all there are

$$1+(n-k)+\binom{n-k}{2}+\cdots+\binom{n-k}{n-k-1}=\sum_{s=0}^{n-k-1}\binom{n-k}{s}$$

admissible $\boldsymbol{\mu}$. Since it is well known that

$$\sum_{s=0}^{m} \binom{m}{s} = (1+1)^m = 2^m,$$

the number of admissible $\boldsymbol{\mu}$ is clearly $2^{n-k}-1$, as claimed. □

2.3. **An Extension to Quadratic Character.** The purpose of this section is to deduce an analog of Theorem 2.3 which holds when the character on the space of half-integral weight forms is an arbitrary quadratic character.

To describe the situation further, we set some notation. As above, let N be an odd positive squarefree integer, and $k \geq 3$ an odd integer. Let ψ be an even quadratic Dirichlet character defined modulo $4N$. Rather than attempt to derive another internal decomposition for $S_{k/2}(4N,\psi)$ as in Theorem 2.1, we shall show that $S^+_{k/2}(4N,\psi)$ and $S^+_{k/2}(4N,1)$ are isomorphic as modules for the Hecke algebra $H(N)$, from which we shall derive the

Theorem 2.6. *Let N be an odd positive squarefree integer, and $k \geq 3$ and odd integer. Let ψ be an even quadratic Dirichlet character defined modulo $4N$. Then*

$$S^+_{k/2}(4N,\psi) \cong \bigoplus_{d|N} \left[\mathfrak{S}^+_{k-1}(2d) \oplus 2\mathfrak{S}^+_{k-1}(d)\right]$$

where the isomorphism is as modules for the Hecke algebra $H(N)$.

The idea behind the desired isomorphism is simple. Since N is squarefree, the character ψ is equal to $\left(\frac{Q}{*}\right)$ for some divisor Q of N with (necessarily, since N is squarefree) $\gcd(Q,4N/Q)=1$. A generalized Atkin-Lehner involution W_Q will act as a "hermitian involution" which commutes with the action of the Hecke operators, and hence provides an isomorphism (as modules for the Hecke algebra) between $S^+_{k/2}(4N,1)$ and $S^+_{k/2}(4N,\left(\frac{Q}{*}\right))$.

While W_Q operators have been defined in [1], [7], and [12], none of these operators is sufficiently general to give a commutation relation with the shift operators (B_d) which are fundamental to our definition of the space of oldforms (and hence of newforms). So we introduce a slightly more general operator with the desired properties. The desired properties (commutation relations with various other operators, hermitian properties, etc.) are completely analogous to properties for the integral weight analog. The proofs are more tedious because of the fussiness of automorphy factors, although there are no significant surprises. Therefore we state the properties and give only the barest of outlines of the proofs.

In defining the W_Q operator, there is no reason to assume that N is squarefree or that the character is quadratic. In fact is is quite revealing not to do so, since we shall see why the situation of quadratic character is special.

Let N be a positive integer, and Q an odd divisor of N with $\gcd(Q,4N/Q)=1$. Let ψ be an even Dirichlet character defined modulo $4N$, and let

$$\gamma_Q = \begin{pmatrix} x & y \\ \frac{4N}{Q}z & Qw \end{pmatrix} \in \Gamma_0(\tfrac{4N}{Q}), \quad \delta_Q = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix}, \quad W_Q = \gamma_Q \delta_Q.$$

For $f \in S_{k/2}(4N, \psi)$ define $\widetilde{W}_Q$ by

$$f \mid \widetilde{W}_Q = f \mid \widetilde{W}_Q(\psi) = \psi_Q(y)\psi_{\frac{4N}{Q}}(x) f \mid \gamma_Q^* \tilde{\delta}_Q$$

where $\gamma_Q^* = \{\gamma_Q, j(\gamma_Q, z)\}$ and $\tilde{\delta}_q = \{\delta_Q, Q^{-1/4}\}$ are elements in the metaplectic cover $\mathcal{G}$ of $GL_2^+(\mathbb{Q})$ (see Shimura [8]), and ψ_Q and $\psi_{\frac{4N}{Q}}$ are the Qth and $\frac{4N}{Q}$th parts of ψ. Here for $\gamma \in \Gamma_0(4)$, $j(\gamma, z)$ is the standard automorphy factor

$$j(\gamma, z) = \frac{\theta(\gamma(z))}{\theta(z)}, \quad \theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}.$$

It is trivial to check that $\widetilde{W}_Q$ is well-defined and that Ueda's $\widetilde{W}_Q$ is a special case of this operator.

Proposition 2.7. *The operator $\widetilde{W}_Q$ maps $S_{k/2}(4N, \psi)$ to $S_{k/2}(4N, \overline{\psi}_Q \psi_{\frac{4N}{Q}} \left(\frac{Q}{*}\right))$.*

This follows easily from the observation that for $\gamma \in \Gamma_0(4N)$,

$$\widetilde{W}_Q(\psi)\gamma^* \widetilde{W}_Q(\psi)^{-1} = (W_Q \gamma W_Q^{-1})^* \{1, \left(\tfrac{Q}{d}\right)\}.$$

Next we have the

Proposition 2.8. *Let $f \in S_{k/2}(4N, \psi)$ then*

1. $f \mid \widetilde{W}_Q(\psi) \mid \widetilde{W}_Q(\overline{\psi}_Q \psi_{\frac{4N}{Q}} \left(\frac{Q}{*}\right)) = \left(\frac{-1}{Q}\right)^{k/2} \overline{\psi}_{\frac{4N}{Q}}(Q)\psi_Q(-1) f.$
2. *If $Q' \mid 4N$ with* $\gcd(Q, \frac{4N}{Q}) = \gcd(Q', \frac{4N}{Q'}) = \gcd(Q, Q') = 1$*, then*

$$f \mid \widetilde{W}_Q(\psi) \mid \widetilde{W}_{Q'}(\overline{\psi}_Q \psi_{\frac{4N}{Q}} \left(\tfrac{Q}{*}\right)) = \overline{\psi_{Q'}(Q)} f \mid \widetilde{W}_{QQ'}(\psi).$$

Proof. The proof proceeds along a standard line, but to check the automorphy factors, one must exercise a little care. First note that if $\chi = \left(\frac{Q}{*}\right)$, then the Q and $4N/Q$-parts of χ are

$$\chi_Q = \begin{cases} \left(\frac{Q}{*}\right) & \text{if } Q \equiv 1 \pmod 4 \\ \left(\frac{-Q}{*}\right) & \text{if } Q \equiv 3 \pmod 4 \end{cases} \qquad \chi_{4N/Q} = \begin{cases} 1 & \text{if } Q \equiv 1 \pmod 4 \\ \left(\frac{-1}{*}\right) & \text{if } Q \equiv 3 \pmod 4 \end{cases}$$

Then, as one considers the cases $Q \equiv \pm 1 \pmod 4$, one must recall that

$$\left(\frac{-1}{Qw}\right)^{1/2} = \left(\frac{-1}{Q}\right)^{1/2} \left(\frac{-1}{w}\right)^{1/2} \varepsilon(Q, w)$$

where

$$\varepsilon(Q,w)=\begin{cases}-1 & \text{if } Q\equiv w\equiv 3 \pmod 4\\ 1 & \text{otherwise}\end{cases}$$

Finally one uses the quadratic reciprocity law for Jacobi symbols. □

Next we have the

Proposition 2.9. *Let $f\in S_{k/2}(4N,\psi)$, with Q as above, and let p be a prime, $p\nmid N$. Then*

$$f\mid \tilde{T}_{k/2,4N,\psi}(p^2)\mid \widetilde{W}_Q(\psi)=\psi_Q(p^2)f\mid \widetilde{W}_Q(\psi)\mid \tilde{T}_{k/2,4N,\overline{\psi}_Q\psi_{\frac{4N}{Q}}\left(\frac{Q}{*}\right)}(p^2)$$

Proof. We sketch the basic idea. Let $\tilde{\tau}=\{\tau,p^{1/2}\}\in\mathcal{G}$, where $\tau=\left(\begin{smallmatrix}1&0\\0&p^2\end{smallmatrix}\right)$. The Hecke operator $\tilde{T}_{k/2}(p^2)$ is described by the action of the double coset $p^{(k/4)-1}\Delta_0(4N)\tilde{\tau}\Delta_0(4N)$. If we write

$$\Delta_0(4N)\tilde{\tau}\Delta_0(4N)=\bigcup_\nu \Delta_0(4N)\tilde{\tau}_\nu,$$

then we need to show that

$$\begin{aligned}f\mid \tilde{T}_{k/2,4N,\psi}(p^2)\mid \widetilde{W}_Q(\psi)&=\sum_\nu \psi(a_\nu)f\mid\tilde{\tau}_\nu\mid \widetilde{W}_Q(\psi)\\&=\psi_Q(p^2)\sum_{\nu'}\overline{\psi}_Q(a_{\nu'})\psi_{\frac{4N}{Q}}(a_{\nu'})\left(\frac{Q}{a_{\nu'}}\right)f\mid \widetilde{W}_Q(\psi)\mid\tilde{\tau}_{\nu'}\\&=\psi_Q(p^2)f\mid \widetilde{W}_Q(\psi)\mid \tilde{T}_{k/2,4N,\overline{\psi}_Q\psi_{\frac{4N}{Q}}\left(\frac{Q}{*}\right)}(p^2)\end{aligned}$$

where if $\tilde{\tau}_\nu=\{\tau_\nu,\varphi_\nu\}\in\mathcal{G}$, a_ν is the $(1,1)$-entry of τ_ν.

In fact, we show that for each ν there is a unique ν' so that

$$\psi(a_\nu)f\mid\tilde{\tau}_\nu\mid\widetilde{W}_Q=\psi_Q(p^2)\overline{\psi}_Q(a_{\nu'})\psi_{\frac{4N}{Q}}(a_{\nu'})\left(\frac{Q}{a_{\nu'}}\right)f\mid\widetilde{W}_Q\mid\tilde{\tau}_{\nu'}$$

Using Shimura's notation [8] for the expression of the double coset as a union of right cosets,

$$\Delta_0(4N)\tilde{\tau}\Delta_0(4N)=\bigcup_{b=0}^{p^2-1}\Delta_0(4N)\alpha_b^*\cup\bigcup_{h=0}^{p-1}\Delta_0(4N)\beta_h^*\cup\Delta_0(4N)\sigma^*,$$

we show that in the correspondence $\tilde{\tau}_\nu\leftrightarrow\tilde{\tau}_{\nu'}$, we can relate $\alpha_b^*\leftrightarrow\alpha_{b'}^*$, $\beta_h^*\leftrightarrow\beta_{h'}^*$ and σ^* with itself. Finally, since $p\nmid 4N$, we may assume that in the definition of γ_Q (W_Q), we have $z\equiv 0 \pmod{p^2}$. □

Next we describe the commutation relations of the shift operator and the $\widetilde{W}_Q$-operator.

Proposition 2.10. *Let M be a positive integer, and let Q be an odd divisor of MN with* $\gcd(Q, 4MN/Q) = 1$*. Then for $f \in S_{k/2}(4N, \psi)$ we have*

$$f \mid B_M \mid \widetilde{W}_Q^{4MN}(\psi\left(\tfrac{M}{*}\right)) =$$
$$\begin{cases} \overline{\psi}_Q(M) f \mid \widetilde{W}_Q(\psi) \mid B_M & \text{if } \gcd(Q,M)=1 \\ \left(\frac{-1}{M}\right)^{k/2} \overline{\psi}_{4NM/Q}(M) M^{-k/2} f \mid \widetilde{W}_{Q/M}^{4N}(\psi) & \text{if } M \mid Q \end{cases}$$

where the superscripts on the $\widetilde{W}$-operators indicate the levels of the forms on which they operate.

Proof. This proof is analogous to the others, however it does require a case analysis based upon the congruence class of M modulo 4. ☐

With these preliminary results in hand, we now address the interaction of the $\widetilde{W}_Q$-operator with oldforms and newforms.

Proposition 2.11. *Let N be a positive integer, Q an odd divisor of N with* $\gcd(Q, 4N/Q) = 1$*, and ψ an even Dirichlet character defined modulo $4N$. Then $\widetilde{W}_Q(\psi)$ takes $S^-_{k/2}(4N, \psi)$ to $S^-_{k/2}(4N, \overline{\psi}_Q \psi_{4N/Q}\left(\frac{Q}{*}\right))$, and in particular takes oldforms to oldforms.*

Proof. It is clear from Proposition 2.9, that $\widetilde{W}_Q$ takes Hecke eigenforms to Hecke eigenforms. Let $f \in S^-_{k/2}(4N, \psi)$. By linearity, it is enough to suppose that for some prime $q \mid N$ either $f \in S_{k/2}(4N/q, \psi)$ (provided ψ is defined modulo $4N/q$), or $f = g \mid B_q$ for some $g \in S_{k/2}(4N/q, \psi\left(\frac{q}{*}\right))$ (provided $\psi\left(\frac{q}{*}\right)$ is defined modulo $4N/q$).

For the first case, write

$$W_Q = W_Q^{4N} = \gamma_Q^{4N}\delta_Q = \begin{pmatrix} x & y \\ \frac{4N}{Q}z & Qw \end{pmatrix}\delta_Q$$

We have two subcases depending upon whether $q \mid Q$ or not. Note that if $q \nmid Q$, then $Q \mid 4N/q$. We now write

$$\gamma_Q^{4N} = \begin{cases} \begin{pmatrix} x & y \\ \frac{4N/q}{Q}(qz) & Qw \end{pmatrix} & \text{if } q \nmid Q \\ \begin{pmatrix} x & y \\ \frac{4N/q}{Q/q}z & \frac{Q}{q}(qw) \end{pmatrix} & \text{if } q \mid Q \end{cases}$$

Notice that if $q \nmid Q$, then γ_Q^{4N} is a valid representative for $\gamma_Q^{4N/q}$ (needed to define $\widetilde{W}_{Q/q}^{4N/q}$), while if $q \mid Q$, then γ_Q^{4N} is a valid representative for $\gamma_{Q/q}^{4N/q}$. Thus we have

$$f \mid \widetilde{W}_Q^{4N}(\psi) = \begin{cases} \kappa f \mid \widetilde{W}_Q^{4N/q}(\psi) & \text{if } q \nmid Q \\ \kappa' f \mid \widetilde{W}_{Q/q}^{4N/q} \mid B_q & \text{if } q \mid Q \end{cases}$$

for constants κ and κ'. In either (sub)case, $f \mid \widetilde{W}_Q^{4N}(\psi) \in S^-_{k/2}(4N, \overline{\psi}_Q\psi_{4N/Q}\left(\frac{Q}{*}\right))$.

In the case that $f = g \mid B_q$, we have by Proposition 2.10 that

$$\begin{aligned} f \mid \widetilde{W}_Q^{4N}(\psi) &= g \mid B_q \mid \widetilde{W}_Q^{4N}(\psi) \\ &= \begin{cases} \overline{\psi}_Q(q) g \mid \widetilde{W}_Q^{4N/q}(\psi\left(\frac{q}{*}\right)) \mid B_q & \text{if } q \nmid Q \\ \left(\frac{-1}{q}\right)^{k/2} \overline{\psi}_{4Nq/Q}(q) q^{-k/2} g \mid \widetilde{W}_{Q/q}^{4N/q}(\psi\left(\frac{q}{*}\right)) & \text{if } q \mid Q, \end{cases} \end{aligned}$$

which again shows that $f \mid \widetilde{W}_Q^{4N}(\psi) \in S^-_{k/2}(4N, \overline{\psi}_Q\psi_{4N/Q}\left(\frac{Q}{*}\right))$ as required. □

Our final proposition provides the last tools needed to complete the proof of Theorem 2.6.

Proposition 2.12. *Let N be a positive integer, Q an odd divisor of N with $(Q, 4N/Q) = 1$, and ψ an even Dirichlet character defined modulo $4N$.*

1. *For $f, g \in S_{k/2}(4N, \psi)$, $\langle f, g\rangle = \langle f \mid \widetilde{W}_Q, g \mid \widetilde{W}_Q\rangle$.*
2. *The operator $\widetilde{W}_Q(\psi)$ takes $S^+_{k/2}(4N, \psi)$ to $S^+_{k/2}(4N, \overline{\psi}_Q\psi_{4N/Q}\left(\frac{Q}{*}\right))$, and in particular takes newforms to newforms.*
3. *If ψ is quadratic and N odd and squarefree, then $S^+_{k/2}(4N, 1)$ and $S^+_{k/2}(4N, \psi)$ are isomorphic as modules for the Hecke algebra.*

Proof. The first statement is obvious, since for all $\tilde{\rho} \in \mathcal{G}$, $\langle f \mid \tilde{\rho}, g\rangle = \langle f, g \mid \tilde{\rho}^{-1}\rangle$. For the second statement, let $f \in S^+_{k/2}(4N, \psi)$ be a newform. By Proposition 2.9, $f \mid \widetilde{W}_Q(\psi)$ is again a simultaneous Hecke eigenform. We need only show that it lies in $S^+_{k/2}(4N, \overline{\psi}_Q\psi_{4N/Q}\left(\frac{Q}{*}\right))$. Let $g \in S^-_{k/2}(4N, \overline{\psi}_Q\psi_{4N/Q}\left(\frac{Q}{*}\right))$. Then by Proposition 2.11, $g \mid \widetilde{W}_Q(\overline{\psi}_Q\psi_{4N/Q}\left(\frac{Q}{*}\right)) \in S^-_{k/2}(4N, \psi)$, and $g \mid \widetilde{W}_Q(\overline{\psi}_Q\psi_{4N/Q}\left(\frac{Q}{*}\right)) \mid \widetilde{W}_Q(\psi) = \kappa g$ with $|\kappa| = 1$. Thus

$$\begin{aligned} \langle f \mid \widetilde{W}_Q(\psi), g\rangle &= \kappa\langle f \mid \widetilde{W}_Q(\psi), g \mid \widetilde{W}_Q(\overline{\psi}_Q\psi_{4N/Q}\left(\tfrac{Q}{*}\right)) \mid \widetilde{W}_Q(\psi)\rangle \\ &= \kappa\langle f, g \mid \widetilde{W}_Q(\overline{\psi}_Q\psi_{4N/Q}\left(\tfrac{Q}{*}\right))\rangle = 0 \end{aligned}$$

as required. For the third statement, since N is odd and squarefree, we may take $\psi = \left(\frac{Q}{*}\right)$ for some divisor Q of N (necessarily satisfying $(Q, 4N/Q) = 1$). Then $\widetilde{W}_Q(1)$ maps $S^+_{k/2}(4N, 1)$ to $S^+_{k/2}(4N, \psi)$. Moreover, by Proposition 2.9, f and $f \mid \widetilde{W}_Q(1)$ have the same Hecke eigenvalues, and since $\widetilde{W}_Q(1)$ is an "involution", the proof is complete. □

3. Decomposition Theorems - Prime Power Level

In this section, we decompose the distinguished subspace $S^\emptyset(4q^m, \psi)$ defined by Ueda [12] in terms of the space of newforms as defined in this paper. We give a decomposition of both the full space of cusp forms $S^\emptyset_{k/2}(4q^m, \psi)$ and of the Kohnen subspace $S^{\emptyset}_{k/2}(4q^m,\psi)_K$ where q is an odd prime, $m \geq 2$, and ψ an even quadratic Dirichlet character defined modulo $4q^m$. We begin with Ueda's definition of $S^{\emptyset}$ using twisting operators. We remind the reader of Ueda's notation [12].

Write $N = 4q^m = M_1 M_2^+$ with $M_1 = 1$ and $M_2^+ = q^m$. Let $\Pi = \{q\}$. Write $S(4q^m,\psi)$ to represent either $S_{k/2}(4q^m,\psi)$ or $S_{k/2}(4q^m,\psi)_K$. While we have defined the subspace of oldforms $S^-_{k/2}(4q^m,\psi)$ in equation (2.1), we need to make sure the same definition makes sense in the Kohnen subspace. That is, we want to make the definition

$$S^-_{k/2}(4N,\psi)_K = \sum_{\substack{q \mid N \\ \mathrm{cond}(\psi) \mid 4N/q}} S_{k/2}(4N/q,\psi)_K + \sum_{\substack{q \mid N \\ \mathrm{cond}(\psi\left(\frac{q}{*}\right)) \mid 4N/q}} S_{k/2}(4N/q,\psi\left(\tfrac{q}{*}\right))_K \mid B_q \tag{3.1}$$

We need only check that $S_{k/2}(4N/q,\psi\left(\frac{q}{*}\right))_K \mid B_q \subset S_{k/2}(4N,\psi)_K$. Clearly there is no issue with the level and character. The only condition to check is that the constraints on the Fourier coefficients which define the Kohnen subspace are met.

Let $g = \sum a(n)x^n \in S_{k/2}(4N/q,\psi\left(\frac{q}{*}\right))_K$, and write $g \mid B_q = \sum b(n)x^n \in S_{k/2}(4N,\psi)$. Since ψ is quadratic, we may write $\psi = \left(\frac{N_0}{*}\right)$ for some positive divisor N_0 of N. Let $\varepsilon = \left(\frac{-1}{N_0}\right)$. Then $g \mid B_q \in S_{k/2}(4N,\psi)_K$ provided $b(n) = 0$ for $\varepsilon(-1)^{\frac{k-1}{2}}n \equiv 2,3 \pmod 4$.

Since $g \in S_{k/2}(4N/q,\psi\left(\frac{q}{*}\right))_K$, and $\psi\left(\frac{q}{*}\right) = \left(\frac{qN_0}{*}\right)$, we know that $a(n) = 0$ if $\left(\frac{-1}{q}\right)\varepsilon(-1)^{\frac{k-1}{2}}n \equiv 2,3 \pmod 4$. Now $b(n) = a(n/q) = 0$ unless $q \mid n$, so write $n = qn_0$. Then $b(n) = a(n_0)$. Since $\left(\frac{-1}{q}\right)q \equiv 1 \pmod 4$, if $\varepsilon(-1)^{\frac{k-1}{2}}n \equiv 2,3 \pmod 4$, then

$$\varepsilon(-1)^{\frac{k-1}{2}}n \equiv \varepsilon(-1)^{\frac{k-1}{2}}n_0 q \equiv \left(\frac{-1}{q}\right)\varepsilon(-1)^{\frac{k-1}{2}}n_0 \equiv 2,3 \pmod 4,$$

so that $a(n_0) = b(n) = 0$ as required.

We define the subspace of oldforms in the Kohnen subspace by equation (3.1), and define the subspace of newforms $S^+_{k/2}(4N,\psi)_K$ as the orthogonal complement of $S^-_{k/2}(4N,\psi)_K$ in $S_{k/2}(4N,\psi)_K$. Note that these subspaces are different than the ones Ueda defines, but it is precisely the point of this section to compare these subspaces by decomposing one in terms of the other.

So with $S(4q^m,\psi)$ representing either the full space of cusp forms or the Kohnen subspace, we let $S^{\pm}(4q^m,\psi)$ represent the corresponding subspaces of old or newforms as defined in and immediately following equations (2.1) and (3.1).

We begin our comparison of the decomposition of $S(4q^m,\psi)$ using Ueda's twisting operators and our notion of newforms. It follows from Propositions 1.5, 1.10, and 1.11 of [12] that for $m \geq 2$,

$$S(4q^m,\psi) = S^\emptyset(4q^m,\psi) \perp S(4q^{m-1},\psi\left(\tfrac{q}{*}\right)) \mid B_q \tag{3.2}$$

where $\perp$ denotes orthogonal sum. Moreover, by definition, we know that

$$S(4q^m,\psi) = S^+(4q^m,\psi) \perp S^-(4q^m,\psi),$$

and from the definitions, $S(4q^{m-1},\psi\left(\frac{q}{*}\right)) \mid B_q \subset S^-(4q^m,\psi)$. Thus $S^+(4q^m,\psi) \subset S^\emptyset(4q^m,\psi)$.

Now by definition,

$$S^-(4q^m,\psi) = S(4q^{m-1},\psi) + S(4q^{m-1},\psi\left(\tfrac{q}{*}\right)) \mid B_q.$$

We compute that

$$S(4q^{m-1},\psi) \cap S(4q^{m-1},\psi\left(\tfrac{q}{*}\right)) \mid B_q = S(4q^{m-2},\psi\left(\tfrac{q}{*}\right)) \mid B_q$$

as follows. First it is obvious that the RHS is contained in the LHS. Now let $f = g \mid B_q \in S(4q^{m-1},\psi)$ with $g \in S(4q^{m-1},\psi\left(\frac{q}{*}\right))$. Since f has level $4q^{m-1}$, by Propositions 1.10 and 1.11, we have that $g \in S(4q^{m-2},\psi\left(\frac{q}{*}\right))$ or $g = 0$, which is what we required.

It follows that

$$S^-(4q^m,\psi) = S(4q^{m-1},\psi\left(\tfrac{q}{*}\right)) \mid B_q \oplus \left[\left(S(4q^{m-2},\psi\left(\tfrac{q}{*}\right)) \mid B_q\right)^\perp \text{ in } S(4q^{m-1},\psi)\right]$$

On the other hand, just as in (3.2) for $m \geq 3$,

$$S(4q^{m-1},\psi) = S^\emptyset(4q^{m-1},\psi) \perp S(4q^{m-2},\psi\left(\tfrac{q}{*}\right)) \mid B_q, \tag{3.3}$$

hence

$$S^-(4q^m,\psi) = S^\emptyset(4q^{m-1},\psi) \oplus S(4q^{m-1},\psi\left(\tfrac{q}{*}\right)) \mid B_q.$$

Thus for $m \geq 3$ we have

$$\begin{aligned} S(4q^m,\psi) &= S^\emptyset(4q^m,\psi) \perp S(4q^{m-1},\psi\left(\tfrac{q}{*}\right)) \mid B_q \\ &= S^+(4q^m,\psi) \perp \left[S^\emptyset(4q^{m-1},\psi) \oplus S(4q^{m-1},\psi\left(\tfrac{q}{*}\right)) \mid B_q\right] \end{aligned} \tag{3.4}$$

If only the last sum in the above expression were an orthogonal sum, then by the uniqueness of orthogonal complements, we could easily deduce the decomposition:

$$S^\emptyset(4q^m,\psi) = S^+(4q^m,\psi) \perp S^\emptyset(4q^{m-1},\psi).$$

However, the sum is not necessarily orthogonal. Nonetheless, we will eventually deduce exactly this identity with $\perp$ replaced by $\oplus$. Unfortunately, the proof is a bit more indirect.

We begin by establishing the above identity as an isomorphism as modules for the Hecke algebra $H(q)$ generated by all Hecke operators $\tilde{T}_{k/2}(p^2)$ with $(p,2q) = 1$.

Theorem 3.1. *Let q be an odd prime, $m \geq 2$ an integer, and ψ an even quadratic Dirichlet character defined modulo $4q^m$. Then as $H(q)$-submodules of $S(4q^m, \psi)$,*

$$S^{\emptyset}(4q^m, \psi) \cong \begin{cases} S^{+}(4q^m, \psi) \oplus S^{\emptyset}(4q^{m-1}, \psi) & \text{if } m \geq 3 \text{ or } m = 2,\ \psi = 1 \\ S^{+}(4q^2, \psi) \oplus S^{+}(4q, \psi) & \text{if } m = 2 \text{ and } \psi = \left(\frac{q}{*}\right). \end{cases}$$

Proof. We must separate the cases $m \geq 3$ and $m = 2$. First consider $m \geq 3$. Recall that $H(q)$ is a commutative semisimple algebra. Since ψ is quadratic, by Lemma 5 of [9] $\tilde{T}_{k/2}(p^2) = \tilde{T}_{k/2,4q^m,\psi}(p^2)$ is hermitian, and by easy computations (see [12]), we can verify that all of the summands in equation (3.4) are $H(q)$-submodules of $S(4q^m, \psi)$. Thus the equalities in equation (3.4) also hold as isomorphisms as $H(q)$-modules:

$$\begin{aligned} S(4q^m, \psi) &\cong S^{\emptyset}(4q^m, \psi) \oplus S(4q^{m-1}, \psi \left(\tfrac{q}{*}\right)) \mid B_q \\ &\cong S^{+}(4q^m, \psi) \oplus S^{\emptyset}(4q^{m-1}, \psi) \oplus S(4q^{m-1}, \psi \left(\tfrac{q}{*}\right)) \mid B_q \end{aligned}$$

Using the semisimplicity of $H(q)$ and canceling like terms yields the result in this case. For $m = 2$, equation (3.3) is not valid. By the conventions in Ueda [12], $S(4q, \psi) = S^{\emptyset}(4q, \psi)$, and $S(4, \psi \left(\frac{q}{*}\right)) \mid B_q$ is not necessarily zero.

As above, we do have that

(3.5)
$$\begin{aligned} S(4q^2, \psi) &= S^{\emptyset}(4q^2, \psi) \oplus S(4q, \psi \left(\tfrac{q}{*}\right)) \mid B_q \\ &= S^{+}(4q^2, \psi) \oplus S(4q, \psi \left(\tfrac{q}{*}\right)) \mid B_q \oplus \left[(S(4, \psi \left(\tfrac{q}{*}\right)) \mid B_q)^{\perp} \text{ in } S(4q, \psi)\right]. \end{aligned}$$

Moreover, these equalities also hold as $H(q)$-module isomorphisms.

If $\psi = 1$, then $S(4, \psi \left(\frac{q}{*}\right)) = 0$, so equation (3.5) and semisimplicity of $H(q)$ yield

$$S^{\emptyset}(4q^2, \psi) \cong S^{+}(4q^2, \psi) \oplus S(4q, \psi) = S^{+}(4q^2, \psi) \oplus S^{\emptyset}(4q, \psi)$$

where $S^{\emptyset}(4q, \psi) = S(4q, \psi)$ by Ueda's conventions.

If $\psi \neq 1$ (and hence $\psi = \left(\frac{q}{*}\right)$), we need to compute $\left[(S(4, 1) \mid B_q)^{\perp} \text{ in } S(4q, \left(\frac{q}{*}\right))\right]$. But

$$S(4q, \left(\tfrac{q}{*}\right)) = S^{+}(4q, \left(\tfrac{q}{*}\right)) \perp S^{-}(4q, \left(\tfrac{q}{*}\right)) = S^{+}(4q, \left(\tfrac{q}{*}\right)) \perp S(4, 1) \mid B_q,$$

so $\left[(S(4, 1) \mid B_q)^{\perp} \text{ in } S(4q, \left(\frac{q}{*}\right))\right] = S^{+}(4q, \left(\frac{q}{*}\right))$.

Thus

$$\begin{aligned} S(4q^2, \left(\tfrac{q}{*}\right)) &\cong S^{\emptyset}(4q^2, \left(\tfrac{q}{*}\right)) \oplus S(4q, 1) \mid B_q \\ &\cong S^{+}(4q^2, \left(\tfrac{q}{*}\right)) \oplus S(4q, 1) \mid B_q \oplus S^{+}(4q, \left(\tfrac{q}{*}\right)), \end{aligned}$$

which yields $S^{\emptyset}(4q^2, \left(\frac{q}{*}\right)) \cong S^{+}(4q^2, \left(\frac{q}{*}\right)) \oplus S^{+}(4q, \left(\frac{q}{*}\right))$ as required. ◻

As a corollary we have

Corollary 3.2. *Let the notation be as above, and put* $\mu = \mathrm{ord}_q(\mathrm{cond}(\psi))$ $(= 0$ *or* $1)$, *then*

$$S^{\emptyset}(4q^m, \psi) \cong \bigoplus_{\ell=\mu}^{m} S^{+}(4q^{\ell}, \psi).$$

Proof. It is immediate from Theorem 3.1 that

$$S^{\emptyset}(4q^m, \psi) \cong \begin{cases} \bigoplus_{\ell=2}^{m} S^{+}(4q^{\ell}, \psi) \oplus S^{\emptyset}(4q, \psi) & \text{if } \psi = 1 \\ \bigoplus_{\ell=2}^{m} S^{+}(4q^{\ell}, \psi) \oplus S^{+}(4q, \psi) & \text{if } \psi = \left(\frac{q}{*}\right) \end{cases}$$

If $\psi = \left(\frac{q}{*}\right)$, we are done. To go further, we recall Ueda's convention [12] that $S^{\emptyset}(4q, \psi) = S(4q, \psi)$. If $\psi = 1$, then $S^{-}(4q, 1) = S(4, 1) = S^{+}(4, 1)$, so $S(4q, 1) = S^{+}(4q, 1) \oplus S^{-}(4q, 1) = S^{+}(4q, 1) \oplus S^{+}(4, 1)$, as required. □

Remark 3.3. **Multiplicities of Newforms**. When $S(4q^m, \psi) = S_{k/2}(4q^m, \psi)_K$, Ueda [12] has established a newform theory and strong multiplicity-one theorem. First, he decomposes the space $S^{\emptyset}$ using twisting operators. In our case, this decomposition is quite simple: $S^{\emptyset} = S^{\emptyset,+} \oplus S^{\emptyset,-}$, where $S^{\emptyset,\pm} = \{f \in S^{\emptyset} : f \mid R_q = \pm f\}$ and R_q is the twisting operator associated to $\left(\frac{*}{q}\right)$. He then decomposes each subspace $S^{\emptyset,\pm}$ into a direct sum of a space of oldforms $\hat{\mathfrak{S}}^{\emptyset,\pm}$ and its orthogonal complement $\mathfrak{S}^{\emptyset,\pm}$ (newforms). In our case $S^{\emptyset,\pm} = \mathfrak{S}^{\emptyset,\pm}$. By Theorem 3.11 of [12], each summand $\mathfrak{S}^{\emptyset,\pm}$ satisfies a strong multiplicity-one theorem. In particular, since

$$S^{\emptyset} = S^{\emptyset,+} \oplus S^{\emptyset,-} = \mathfrak{S}^{\emptyset,+} \oplus \mathfrak{S}^{\emptyset,-} \cong \bigoplus_{\ell=\mu}^{m} S^{+}(4q^{\ell}, \psi),$$

we must have that newforms in the spaces $S^{+}(4q^{\ell}, \psi)$ occur with multiplicity one or two. This is analogous to our Theorem 2.3 which states a more precise result for the full space of cusp forms having squarefree level.

We now show that $S^{\emptyset}(4q^{m-1}, \psi) \subset S^{\emptyset}(4q^m, \psi)$ (for $m \geq 3$). To do so, we first need to compute the adjoint of the operator U_q. The proposition below is stated for the full space of cusp forms. The action on the Kohnen subspace is the same, as we discuss following the proof.

Proposition 3.4. *Let q be an odd prime and N a positive integer. Suppose that $q \mid N$, and ψ is an even Dirichlet character defined modulo $4N$. Let $U_q^* : S_{k/2}(4N, \psi\left(\frac{q}{*}\right)) \to S_{k/2}(4N, \psi)$ be the adjoint of the U_q operator. Then for $g \in S_{k/2}(4N, \psi\left(\frac{q}{*}\right))$,*

$$g \mid U_q^* = q^{k/2} g \mid B_q \mid \mathrm{Tr}_{4N}^{4Nq}(\psi)$$

where B_q is the shift operator and Tr *is the trace operator.*

Proof. Let $\Gamma_1 = \Gamma_1(4N)$, $\Delta_1 = \Delta_1(4N)$, $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q \end{smallmatrix}\right)$ and $\xi = [\alpha, q^{1/4}] \in \mathcal{G}$. Then $U_q = \Delta_1 \xi \Delta_1$. By Lemma 1.9 of [11] (using Proposition 1.1 and equation (1.14) of [8]), we have $U_q^* = \Delta_1 \xi' \Delta_1$, where $\xi' = [\alpha', q^{1/4}] \in \mathcal{G}$, $\alpha' = \left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

If $\alpha_\mu = \left(\begin{smallmatrix} 1 & 0 \\ 4N\mu & 1 \end{smallmatrix}\right)$, then since $q \mid N$, we have

$$\Gamma_1 = \Gamma_1(4N) = \cup_{\mu=1}^{q} \Gamma_1(4Nq)\alpha_\mu \text{ and } \Gamma_1 \alpha \Gamma_1 = \cup_{\mu=1}^{q} \Gamma_1 \alpha \alpha_\mu.$$

Again by Proposition 1.1 and equation (1.14) of Shimura [8]

$$\Delta_1(4N)\xi'\Delta_1(4N) = \cup_{\mu=1}^{q} \Delta_1(4N)\xi'\alpha_\mu^*$$

where $\alpha_\mu^* = [\alpha_\mu, j(\alpha_\mu, \tau)] \in \Delta_1 \subset \mathcal{G}$.

So for $g \in S_{k/2}(4N, \psi\left(\frac{q}{*}\right))$, we have

$$\begin{aligned} g \mid U_q^* = g \mid \Delta_1 \xi' \Delta_1 &= \det(\alpha')^{k/4-1} \sum_{\mu=1}^{q} g \mid \xi' \alpha_\mu^* \\ &= q^{k/2-1} \sum_{\mu=1}^{q} g \mid B_q \mid \alpha_\mu^* = q^{k/2-1} \sum_{\mu=1}^{q} \overline{\psi}(d_\mu)(g \mid B_q) \mid \alpha_\mu^* \end{aligned}$$

where $d_\mu = 1$ is the d-entry of α_μ

$$= q^{k/2-1}[\Gamma_1(4N) : \Gamma_1(4Nq)] g \mid B_q \mid \mathrm{Tr}_{4N}^{4Nq}(\psi) = q^{k/2} g \mid B_q \mid \mathrm{Tr}_{4N}^{4Nq}(\psi)$$

as required. □

We note that the same expression makes sense as well for the Kohnen subspace, provided N is odd so the space is defined. Apriori, the adjoint of the U_q operator restricted to the Kohnen subspace is the orthogonal projection of the adjoint U_q^* into the Kohnen subspace, but we have already observed that the B_q operator preserves the Kohnen subspace, and since the trace operator is simply the adjoint of the inclusion map, it does as well.

Proposition 3.5. *Let $m \geq 2$, and $f \in S^\emptyset = S^\emptyset(4q^m, \psi)$. Then $f \mid U_q = 0$.*

Proof. Recall that $S^\emptyset = S^{\emptyset,+} \oplus S^{\emptyset,-}$ where $S^{\emptyset,\pm} = \{f \in S^\emptyset : f \mid R_q = \pm f\}$, and R_q is the twisting operator associated to $\left(\frac{*}{q}\right)$. Clearly, it is enough to verify the result for $f \in S^{\emptyset,\pm}$. For such an $f = \sum a_n x^n$, $f \mid R_q = \pm f$ means that $a_n = 0$ whenever $q \mid n$. Thus $f \mid U_q = \sum a_{nq} x^n = 0$. □

Proposition 3.6. *For $m \geq 3$, $S^\emptyset(4q^{m-1}, \psi) \subset S^\emptyset(4q^m, \psi)$.*

Proof. Let $f \in S^{\emptyset}(4q^{m-1},\psi) \subset S(4q^m,\psi)$. Then

$$\begin{aligned} f \in S^{\emptyset}(4q^m,\psi) &\iff \langle f, g \mid B_q\rangle_{4q^m,\psi} = 0 \text{ for all } g \in S(4q^{m-1},\psi\left(\tfrac{q}{*}\right)) \\ &\iff \langle f, g \mid B_q \mid \mathrm{Tr}^{4q^m}_{4q^{m-1}}\rangle_{4q^{m-1},\psi} = 0 \text{ for all } g \in S(4q^{m-1},\psi\left(\tfrac{q}{*}\right)) \\ &\iff \langle f, g \mid U_q^*\rangle = 0 \text{ for all } g \\ &\iff \langle f \mid U_q, g\rangle = 0 \text{ for all } g \end{aligned}$$

which is true by the previous proposition. □

Corollary 3.7. *For $m \geq 3$, $S^{\emptyset}(4q^m,\psi) = S^+_{k/2}(4q^m,\psi) \oplus S^{\emptyset}(4q^{m-1},\psi)$.*

Proof. We have already verified that both summands are contained in the RHS. Moreover, $S^+_{k/2}(4q^m,\psi) \cap S^{\emptyset}(4q^{m-1},\psi) = 0$ since $S^{\emptyset}(4q^{m-1},\psi) \subset S(4q^{m-1},\psi) \subset S^-_{k/2}(4q^m,\psi)$. Thus the RHS is a summand of the LHS. But by Theorem 3.1, both sides are isomorphic as $H(q)$-modules, and hence as finite dimensional vector spaces from which the assertion follows. □

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