# MAXIMAL ORDERS IN CENTRAL SIMPLE ALGEBRAS AND BRUHAT–TITS BUILDINGS

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Abstract. We study the affine building for $SL_n$ over a local field and give a characterization of distance involving Hecke operators. For $n = 3$ we give an explicitly computable distance formula. We use this local information to show that the class number of a maximal order in a central simple algebra of dimension $n^2$ over a number field $K$ is equal to the number of orbits of a group of isometries (related to the unit group of the maximal order) acting on a Bruhat-Tits building for $SL_n(K)$. This generalizes results of Serre and Vignéras who considered the quaternion case in which the Bruhat-Tits building is a tree.

## 1. Introduction

We begin with a summary of the quaternion case. Let $F$ be a non-archimedean local field with valuation ring $R$, $A$ the quaternion algebra $M_2(F)$, and $V$ a 2-dimensional vector space over $F$. If we identify $A$ with $\mathrm{End}_F(V)$, then it is well known that every maximal order in $A$ is of the form $\Lambda = \mathrm{End}_R(L)$ where $L$ is a free $R$-module (lattice) of rank 2. Moreover, any two maximal orders are conjugate by an element of $A^\times$, and if $M$ is another lattice of rank 2, then $\mathrm{End}_R(L) = \mathrm{End}_R(M)$ if and only if $L = \lambda M$ for some nonzero element $\lambda$ of $F^\times$. Thus the maximal orders in $A$ are in 1–1 correspondence with classes of lattices $[L]$, where $[L] = [M]$ if and only if $L = \lambda M$ as above.

Given two lattices $L$ and $M$ on $V$, Serre [7, p 70] defines the distance between them using elementary divisors: If $\pi$ is a uniformizing parameter for $R$, then by the elementary divisor theorem there is a basis $\{e_1, e_2\}$ of $L$ such that $\{\pi^a e_1, \pi^b e_2\}$ is a basis for $M$. The distance between $L$ and $M$ is defined to be the absolute value $|a-b|$. It is trivial to observe that the distance is unchanged by replacing a lattice with another in the same class; hence the distance function can be viewed as a function on classes of lattices, or equivalently on the set $X$ of maximal orders in $A$. We can view the elements of $X$ as the vertices of a graph in which two maximal orders are connected by an edge if they are distance one apart. Serre shows that $X$ is a tree and that it is $(q + 1)$-regular, where $q$ is the cardinality of the residue class field $R/\pi R$.

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Vignéras [9] continues the examination of the quaternion case by shifting to a consideration of quaternion algebras over the ground field $\mathbb{Q}$. Let $A$ be a positive definite rational quaternion algebra, $\Lambda$ a maximal order in $A$, and let $p$ be a prime which splits in $A$, i.e., for which $A_p = A \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$. Serre's tree is defined in the local quaternion algebra $A_p$. Vignéras shifts the focus to the global algebra by means of the local-global correspondence for orders: We let $X$ be the set of maximal orders $\Gamma$ in $A$ for which $\Gamma_q = \Lambda_q$ for all primes $q \neq p$. The elements of $X$ are in 1–1 correspondence with the maximal orders in $A_p$, and we may transport the distance function for maximal orders in $A_p$ to $A$ by designating the distance between two global orders $\Gamma$ and $\Gamma'$ to be the distance between their localizations at $p$: $\Gamma_p$ and $\Gamma'_p$. As in the case of Serre, we can make the set $X$ into a graph by designating the maximal orders as vertices and by placing edges between vertices which are distance one apart. Setting $\Lambda^{(p)} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$, we see that there is a natural action of the unit group $\Lambda^{(p)\times}$ on the graph $X$. Under this action, Vignéras shows that the class number of $\Lambda$ (and hence of $A$) is equal to the number of orbits of $\Lambda^{(p)\times}$ in $X$.

Finally, we note that the classical Hecke operators (for $SL_2(\mathbb{Z})$) make an appearance at this stage. Here we remark only that it is not a coincidence that the $(p+1)$-regularity of the graph $X$ is equal to the degree of the Hecke operator $T_p$. We shall explore this interaction carefully in our generalization of these results.

In this paper, we extend the above results to arbitrary central simple algebras (of dimension $n^2$) over a number field $K$. In general, our interest is in $n \geq 3$, both because the quaternion case generalizes formally to the number field setting, and also because there is a fundamental difference between the structure which results on the set $X$ of maximal orders when $n = 2$ or when $n \geq 3$. The structure one obtains is always isomorphic to the Bruhat-Tits building for $SL_n(F)$, $F$ a non-archimedean local field. However, in the case $n = 2$ (the quaternion case), the building is a tree, while in general it is isomorphic to an $(n-1)$-dimensional simplicial complex on the set $X$. The fundamental difference between the quaternion and the higher rank setting is most easily visible in the difference between the Coxeter diagrams (see [1, p 148]), which we shall discuss in more detail later.

This paper consists of several parts. The first part establishes a correspondence between the set of maximal orders in the central simple algebra and the vertices of the Bruhat-Tits building for $SL_n(K)$. It will once again turn out that we may think interchangeably of the vertices in the building as maximal orders or classes of lattices. An important consequence is that we are able to characterize the distance between classes of lattices (in terms of elementary divisors) in a way which is compatible with the natural distance function on the building. For $n = 3$ we give an explicit distance formula. Next we establish the regularity of the underlying graph by computing the valence of each vertex in terms of the degree of rank $n$ Hecke operators. For ranks $n = 3, 4, 5$, we compute the valence explicitly. Finally, we show that the class number of maximal orders is equal to the number of orbits of a certain unit group acting on the set $X$ of maximal orders. The proof is an adelic argument requiring an application of Eichler's norm theorem which characterizes principal ideals in the algebra in terms of their norms.

## Preliminaries

Let $K$ be a number field with ring of integers $\mathcal{O}$. For a prime $\mathfrak{P}$ (finite or infinite) of $K$, let $K_{\mathfrak{P}}$ denote its completion with respect to the usual $\mathfrak{P}$-adic valuation, and when $\mathfrak{P}$ is finite, let $\mathcal{O}_{\mathfrak{P}}$ denote the valuation ring of $K_{\mathfrak{P}}$. For any Dedekind domain $R$, $\mathcal{O} \subset R \subset K$, let $R_{\mathfrak{P}}$ denote the closure of $R$ in $K_{\mathfrak{P}}$. Throughout, let $\pi$ denote a uniformizing parameter for $\mathcal{O}_{\mathfrak{P}}$.

For later convenience in applying the Eichler norm theorem (Theorem 5.2), we want to choose a Dedekind domain $R$, $\mathcal{O} \subset R \subset K$ with the property that $R$ is a PID and yet with the property that for all but a finite number of primes $\mathfrak{P}$ of $K$, $R_{\mathfrak{P}} = \mathcal{O}_{\mathfrak{P}}$. This choice also obviates the need to restrict to number fields of class number one. In fact, it is convenient to construct a Dedekind domain containing $\mathcal{O}$ which has *strict* class number one. Let $S$ be a finite set of primes of $K$ including all the infinite ones. Put $\mathcal{O}^{(S)} = \{x \in K \mid x \in \mathcal{O}_{\mathfrak{P}} \text{ for all } \mathfrak{P} \notin S\}$. Note that $\mathcal{O}^{(S)}$ is not a localization of $\mathcal{O}$; rather it is the intersection of all localizations $\mathcal{O}_{(\mathfrak{P})}$ for $\mathfrak{P} \notin S$. In particular, when $K = \mathbb{Q}$ and $S$ consists of the infinite prime, $\mathcal{O}^{(S)} = \mathbb{Z}$.

**Lemma 1.1.** *There exists a finite set of primes $S$, which contains all the infinite primes, and for which $\mathcal{O}^{(S)}$ is a PID. Moreover, the set $S$ can be chosen in such a way that $\mathcal{O}^{(S)}$ has strict class number one.*

*Proof.* For any set $S$ of the required form, $\mathcal{O}^{(S)}$ is a Dedekind domain (see Chapter II of [3]), so we need only show that $\mathcal{O}^{(S)}$ has strict class number one for some set $S$. Let $S$ consist of the infinite primes together with one prime from each strict ideal class of $\mathcal{O}$. Let $\mathcal{I}$ be a fractional $\mathcal{O}^{(S)}$-ideal. Then $\mathcal{I}$ is a fractional $\mathcal{O}$-ideal, and so can be written as $\mathcal{I} = \mathfrak{P}_1^{e_1}\mathfrak{P}_2^{e_2}\cdots\mathfrak{P}_r^{e_r}\cdot\mathfrak{Q}$ where the $\mathfrak{P}_i \in S$ and where $\mathfrak{Q}$ is a fractional $\mathcal{O}$-ideal relatively prime to the primes in $S$. It is easy to see that $\mathcal{I}\mathcal{O}^{(S)} = \mathfrak{Q}\mathcal{O}^{(S)}$ since $\mathfrak{P}\mathcal{O}^{(S)} = \mathcal{O}^{(S)}$ for any $\mathfrak{P} \in S$. Finally by the assumptions on the set $S$, $\mathfrak{Q}$ is in the same strict ideal class as some prime $\mathfrak{P}$ in $S$, so $\mathfrak{Q} = \mathfrak{P}\alpha$ for some totally positive $\alpha \in K^\times$. Thus $\mathcal{I} = \mathcal{I}\mathcal{O}^{(S)} = \mathcal{O}^{(S)}\alpha$, whence $\mathcal{O}^{(S)}$ has strict class number one. □

*Remark 1.2.* Observe that $\mathcal{O}_{\mathfrak{P}}^{(S)} = \mathcal{O}_{\mathfrak{P}}$ for all $\mathfrak{P} \notin S$, and that for any finite set $T \supset S$, $\mathcal{O}^{(T)} \supset \mathcal{O}^{(S)}$ and $\mathcal{O}^{(T)}$ also has strict class number one. Finally, we note that the reader will suffer little loss of understanding in assuming throughout that $K = \mathbb{Q}$ and $\mathcal{O} = \mathcal{O}^{(S)} = \mathbb{Z}$.

Let $A$ be a central simple algebra of dimension $n^2$ over $K$, $n \geq 3$. For any Dedekind domain $R$ with $\mathcal{O} \subset R \subset K$, we call a subset $\Lambda \subset A$ an $R$-order if $\Lambda$ is a subring of $A$ having the same multiplicative identity as $A$, and $\Lambda$ is a finitely generated $R$-module such that $K \otimes_R \Lambda \cong A$. When $R$ is a PID, an order is a free $R$-module of rank $n^2$.

Throughout we shall keep $R = \mathcal{O}^{(S)}$ with $S$ fixed and chosen as above so that all $R$-orders are free $R$-modules. Fix a maximal $R$-order $\Lambda$ in $A$, and choose a prime $\mathfrak{P} \notin S$

which is unramified in $A$ (see §32 of [4]). Then $A_{\mathfrak{P}} \cong M_n(K_{\mathfrak{P}})$ and by Theorem 17.3 of [4], $\Lambda_{\mathfrak{P}}$ is conjugate to $M_n(\mathcal{O}_{\mathfrak{P}})$. We may assume without loss of generality that $A_{\mathfrak{P}}$ is identified with $M_n(K_{\mathfrak{P}})$ in such a way that $\Lambda_{\mathfrak{P}} = M_n(\mathcal{O}_{\mathfrak{P}})$. In particular, let $V$ be an $n$-dimensional vector space over $K_{\mathfrak{P}}$ and $L$ a free $\mathcal{O}_{\mathfrak{P}}$-module of rank $n$. Then with respect to some basis, we assume that $A_{\mathfrak{P}} = \mathrm{End}_{K_{\mathfrak{P}}}(V)$ and $\Lambda_{\mathfrak{P}} = \mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(L)$.

## 2. The Bruhat-Tits Building for $SL_n(K_{\mathfrak{P}})$

In this section we characterize the Bruhat-Tits Building for $SL_n(K_{\mathfrak{P}})$ as an $(n-1)$-dimensional simplicial complex on the set of maximal orders in $A_{\mathfrak{P}}$, and give a characterization of the distance between maximal orders (vertices in the building) in terms of the elementary divisors of the lattices whose classes determine the orders.

To begin, recall that the affine building $\Delta_n$ for $SL_n(K_{\mathfrak{P}})$ has type $\widetilde{A}_{n-1}(K_{\mathfrak{P}})$ and can be realized as an $(n-1)$-dimensional simplicial complex on the set of classes of lattices as follows (for complete details see [1, Chapter V.8] or [6, Chapter 9.2]): The type of the building specifies a certain Coxeter diagram, which implicitly specifies a collection of positive integers $m_{ij}$ used to construct the $(n-1)$-simplices. Associated to the Coxeter diagram is a Coxeter group, $W$, which is given by generators and relations as

$$W = \langle r_i \mid r_i^2 = (r_i r_j)^{m_{ij}} = 1 \text{ for } 1 \le i, j \le n \rangle.$$

The building is comprised of apartments, each of which is a Coxeter complex whose geometric realization is a tesselation of $\mathbb{R}^{n-1}$ by $(n-1)$-simplices called chambers. The $(n-1)$-simplices which are used in the tesselation have dihedral angles $\pi/m_{ij}$ where the integers $m_{ij}$ are specified by the corresponding Coxeter diagram. For example, in the building for $SL_3(K_{\mathfrak{P}})$, each apartment can be visualized as the Euclidean plane tesselated by equilateral triangles, and for $SL_4(K_{\mathfrak{P}})$, the apartments can be visualized as Euclidean 3-space tesellated by tetrahedra with two dihedral angles $\pi/3$ and the other two $\pi/2$.

For most of the important results, attention may be restricted to apartments, so we now give a more careful description of them. The Coxeter group $W$ acts simply-transitively on the set of chambers in an apartment. For convenience, we therefore specify a fundamental chamber on which $W$ acts to generate the apartment. To characterize this notion in terms of classes of lattices we let, as in the previous section, $V$ be an $n$-dimensional vector space over $K_{\mathfrak{P}}$. For $L$, $M$ lattices on $V$ (free $\mathcal{O}_{\mathfrak{P}}$-modules of rank $n$), we say that $L$ and $M$ are in the same class if $L = \lambda M$ for some $\lambda \in K_{\mathfrak{P}}^\times$, and we denote the class of $L$ by $[L]$. As a matter of notation, if the lattice $L$ has $\mathcal{O}_{\mathfrak{P}}$-basis $\{v_1, \dots, v_n\}$, we will often write $[v_1, \dots, v_n]$ for $[L]$. Fix a basis $\{v_1, \dots, v_n\}$ for $V$. From [1, p 137], the fundamental chamber $C$ of $\Delta_n$ can be taken to be the $(n-1)$-simplex with vertices $[v_1, \dots, v_{i-1}, \pi v_i, \dots, \pi v_n]$, $i = 1, 2, \dots, n$. By [1, p 148], the fundamental apartment $\Sigma$ of $\Delta_n$ (the apartment generated by the action of $W$ on $C$) has vertices $[\pi^{a_1} v_1, \dots, \pi^{a_n} v_n]$ with the $a_i \in \mathbb{Z}$.

Our principle interest is the study of maximal orders in central simple algebras, so we first show that the vertices of our building $\Delta_n$ may be taken to be maximal orders in the local algebra $A_{\mathfrak{P}}$. From the previous section, we know that all maximal orders in $A_{\mathfrak{P}}$ are of the form $\mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(L)$ for some lattice $L$ on $V$. First we determine all possible redundancy in this characterization.

**Proposition 2.1.** *Let $\Gamma_1 = \mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(L_1)$ and $\Gamma_2 = \mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(L_2)$ be two maximal orders in $A_{\mathfrak{P}}$. Then $\Gamma_1 = \Gamma_2$ if and only if $L_1 = \alpha L_2$ for some $\alpha \in K_{\mathfrak{P}}^{\times}$.*

*Proof.* That $\mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(L) = \mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(\alpha L)$ is obvious since every element of $\mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(L)$ extends to a unique element of $\mathrm{End}_{K_{\mathfrak{P}}}(V)$ by linearity. We establish the converse by showing the contrapositive. By the elementary divisor theorem, there is an $\mathcal{O}_{\mathfrak{P}}$-basis $\{e_1, e_2, \ldots, e_n\}$ of $L_1$ and there are rational integers $a_1, \ldots, a_n$ such that $\{\pi^{a_1}e_1, \pi^{a_2}e_2, \ldots, \pi^{a_n}e_n\}$ is an $\mathcal{O}_{\mathfrak{P}}$-basis of $L_2$. Now $L_2 = \alpha L_1$ iff $L_2 = \pi^a L_1$ for some $a$, so to begin the proof, we assume that $L_2 \neq \pi^a L_1$, and hence that there is an index $j$ for which $a_j \neq a_1$. Consider the element $\varphi \in \mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(L_1)$ induced by $\varphi(e_j) = e_1$, $\varphi(e_1) = e_j$ and $\varphi(e_i) = e_i$ for all $i \neq 1, j$. Then $\varphi(L_2) \not\subset L_2$ which completes the proof. $\square$

We see from Proposition 2.1 that there is a 1–1 correspondence between classes of lattices on $V$ and maximal orders in $A_{\mathfrak{P}}$, so we will speak of the vertices of our building interchangeably as maximal orders or as classes of lattices. Also from the proposition, we see that whenever convenient, given two maximal orders $\mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(L_1)$ and $\mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(L_2)$, we may assume that $L_1 \subset L_2$. Hence by the elementary divisor theorem, we may assume that there is an $\mathcal{O}_{\mathfrak{P}}$-basis $\{e_1, e_2, \ldots, e_n\}$ of $L_2$ and there are rational integers $0 \leq a_1 \leq \ldots \leq a_n$ such that $\{\pi^{a_1}e_1, \pi^{a_2}e_2, \ldots, \pi^{a_n}e_n\}$ is an $\mathcal{O}_{\mathfrak{P}}$-basis of $L_1$.

We want to discuss the notion of distance in the building $\Delta_n$ and to show that the natural distance can be interpreted in terms of the elementary divisors of the lattices whose endomorphism rings are the maximal orders comprising the vertices in our building. As an end result, we shall show that the number of vertices at distance one from a given vertex corresponds to the degree of a certain Hecke operator (Theorem 3.3), and that the class number of a maximal order in the global algebra $A$ is the number of orbits of a certain group acting as isometries on the building (Theorem 5.3).

First we discuss distance in the general framework of a building. Recall that two chambers in a building are adjacent if they share a codimension-one face (a panel, which determines a unique wall). Within an apartment, we can be more explicit. Since the apartments of a building are Coxeter complexes, the notion of adjacency within an apartment can be described via the Coxeter group as follows (see [6, p 10]): The chambers in an apartment are labeled by the elements of the Coxeter group $W = \langle r_i \mid r_i^2 = (r_i r_j)^{m_{ij}} = 1 \text{ for } 1 \leq i, j \leq n\rangle$ in such a way that $i$-adjacency is defined by $w \underset{i}{\sim} wr_i$. Recall that $W$ acts simply-transitively by left translation on the chambers in an apartment, and it is clear that this action preserves $i$-adjacency. For $SL_3(K_{\mathfrak{P}})$, the apartment would be labelled (starting from an arbitrary chamber $g \in W$ as in Figure 2.1. Notice that the lines of different shades and thicknesses represent the three types of adjacency.

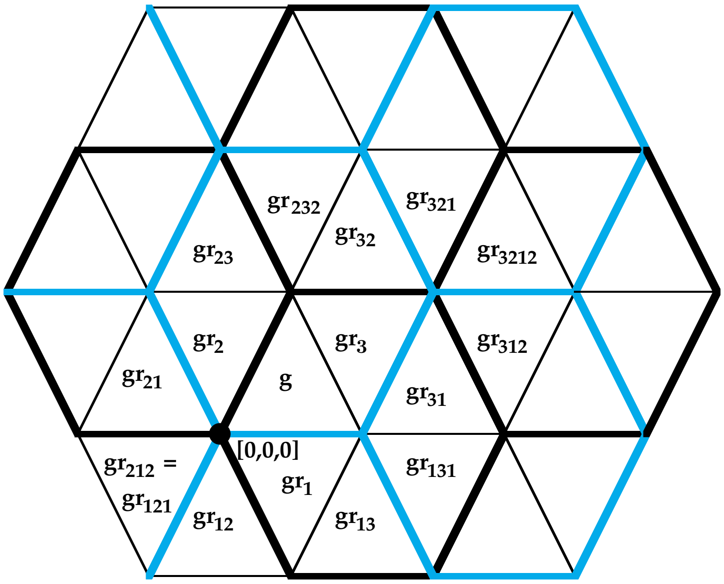


FIGURE 2.1

A gallery in a building is a sequence of chambers (equilateral triangles for $SL_3(K_{\mathfrak{P}})$), $C_0, C_1, \ldots, C_d$ in which consecutive chambers $C_{i-1}$ and $C_i$ ($i = 1, \ldots d$) are adjacent. The length of the gallery is $d$. The distance between two chambers is defined to be the length of any minimal gallery between the two chambers. Finally, since two distinct adjacent chambers are separated by a unique wall, given any minimal gallery $C_0, \ldots, C_d$, the walls crossed by the gallery are distinct and are precisely the walls separating $C_0$ from $C_d$. In particular, the distance $d$ between the two chambers $C_0$ and $C_d$ is equal to the number of walls separating them (see [1, p 73]). Since any two chambers are contained in an apartment $\Sigma$ [6, Corollary 3.7], and any minimal gallery between those chambers lies within $\Sigma$, [6, Theorem 3.8], the distance between chambers can be computed within any apartment which contains both chambers. Thus we can rephrase the notion of distance again in terms of the labeling of the apartment by elements of the Coxeter group $W$. It is clear that the distance from chamber $g$ to the chamber $gw$ (where $w$ is a reduced word in $W$) is the length of $w$.

Now we wish to give a formulation of distance which takes advantage of the building's characterization as a simplicial complex on classes of lattices and use the fact that we have restricted our attention to an apartment. Recall that we have fixed a basis $\{v_1, \ldots, v_n\}$ for $V$. The fundamental chamber $C$ of $\Delta_n$ is the $(n-1)$-simplex with vertices $[v_1, \ldots, v_{i-1}, \pi v_i, \ldots, \pi v_n]$, $i = 1, 2, \ldots, n$, and the fundamental apartment $\Sigma$ of $\Delta_n$ has vertices $[\pi^{a_1} v_1, \ldots, \pi^{a_n} v_n]$ with the $a_i \in \mathbb{Z}$. Since the basis is fixed, we simplify the typography further by writing $[a_1, \ldots, a_n]$ for $[\pi^{a_1} v_1, \ldots, \pi^{a_n} v_n]$. Then since we are concerned only with classes of lattices we observe that each class $[a_1, \ldots, a_n]$ has a unique representative of the form $[0, b_2, \ldots, b_n]$ (with $b_i = a_i - a_1$).

With this notation we again get a labelling of the apartment $\Sigma$, this time by a labelling of its vertices. For $SL_3(K_{\mathfrak{P}})$, this is shown in Figure 2.2.

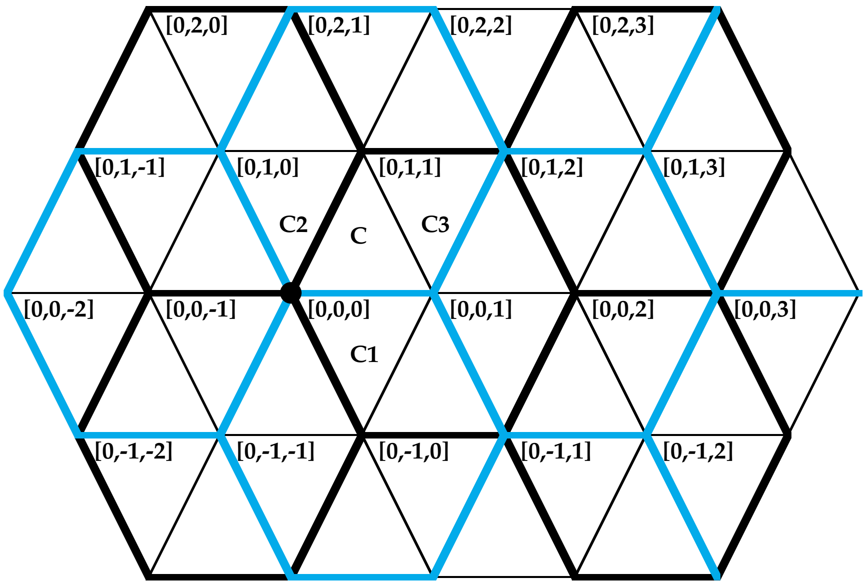


FIGURE 2.2

*Remark 2.2.* In practice we shall choose the basis $\{v_1, \ldots, v_n\}$ of $V$ to facilitate the calculation of distance. For example, to find the distance between vertices $v$ and $w$, first represent $v = [L]$ and $w = [M]$ where $L$ and $M$ are two lattices on $V$. By the elementary divisor theorem, there exists a basis $\{v_1, \ldots, v_n\}$ of $L$ (and hence of $V$) and integers $a_1 \leq \cdots \leq a_n$ so that $L = \oplus \mathcal{O}_{\mathfrak{P}} v_i$ and $M = \oplus \mathcal{O}_{\mathfrak{P}} \pi^{a_i} v_i$. The collection $\{\pi^{a_1}, \ldots, \pi^{a_n}\}$ is called the set of elementary divisors of $M$ in $L$, denoted $\{L : M\}$.

The basis $\{v_1, \ldots, v_n\}$ generates an apartment $\Sigma$ containing $v$ and $w$ as above. For any other apartment $\Sigma'$ containing $v$ and $w$, there is an isomorphism (preserving adjacency) which fixes $v$ and $w$ [1, IV.1]. In particular, the distance between $v$ and $w$ is independent of the apartment in which we consider $v$ and $w$, so we may do our calculations within $\Sigma$. Since the vertices $v$ and $w$ are determined in $\Sigma$ by the elementary divisors $\{L : M\}$ (i.e., $v = [0, \ldots, 0]$ and $w = [a_1, \ldots, a_n]$), the distance between $v$ and $w$ is completely determined by the elementary divisors $\{L : M\}$.

In the building $\Delta_n$, the chambers correspond to maximal flags of lattices on $V$ (see [6, Chapter 9.2]):

$$\pi L_0 \subset L_{n-1} \subset L_{n-2} \subset \cdots \subset L_0$$

where the vertices of the chamber are $[L_0]$, $[L_1]$, ... , $[L_{n-1}]$, and $\pi$ is a uniformizing parameter for $\mathcal{O}_{\mathfrak{P}}$. Our goal is to define the distance between vertices in the building. To do so, we need to connect the notion of distance described in terms of galleries with that of the elementary divisors of the lattices. We proceed as follows. As in Chapter 9.2 of [6], we can realize the Coxeter group $W$ as a group of reflections generated by reflections in the codimension-1 faces of the fundamental chamber. In particular, we can define reflections $S_i$, $1 \leq i \leq n$ in terms of their actions on elementary divisors (actually on classes of lattices):

$$S_i([\alpha_1, \ldots, \alpha_n]) = [\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \alpha_{i+2}, \ldots \alpha_n] \text{ for } 1 \leq i \leq n-1, \text{ and}$$
$$S_n([\alpha_1, \ldots, \alpha_n]) = [\alpha_n - 1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_1 + 1].$$

The first $n-1$ reflections merely interchange the $i^{\text{th}}$ and $i+1^{\text{st}}$ coordinates. In terms of our standard representation of vertices, we have:

$$S_1([0, \beta_2, \ldots, \beta_n]) = [0, -\beta_2, \beta_3 - \beta_2, \ldots \beta_n - \beta_2],$$
$$S_i([0, \beta_2, \ldots, \beta_n]) = [0, \beta_2, \ldots, \beta_{i-1}, \beta_{i+1}, \beta_i, \beta_{i+2}, \ldots, \beta_n] \text{ for } 2 \leq i \leq n-1, \text{ and}$$
$$S_n([0, \beta_2, \ldots, \beta_n]) = [0, \beta_2 - \beta_n + 1, \ldots, \beta_{n-1} - \beta_n + 1, 2 - \beta_n].$$

To deduce the action on chambers, let $C$ denote the fundamental chamber with vertices described by the flag

$$\pi L_0 \subset L_{n-1} = [0, 1, \ldots, 1] \subset L_{n-2} = [0, 0, 1, \ldots, 1] \subset$$
$$\cdots \subset L_1 = [0, \ldots, 0, 1] \subset L_0 = [0, \ldots, 0].$$

It is then straightforward to verify the action of each $S_i$ on the chamber $C$. In particular, we see that $S_i$ fixes $L_k$ for $k \neq n - i$. That is, each $S_i$ fixes $n-1$ of the $n$ vertices of the chamber $C$, and so $C$ and its image are reflections across a codimension-1 face of $C$. Now the $S_i$ satisfy all the same relations as the generators of the Coxeter group $W$ [6], so we associate the $S_i$ with the corresponding $r_i$ so that $i$-adjacency denoted by $g \underset{i}{\sim} gr_i$ is compatible with the $i$-adjacency of $C$ and $S_i(C)$.

For later convenience, we now denote $S_i$ by $S_{r_i}$, where $r_i$ is the corresponding generator of $W$. Moreover, if $w = r_{i_1} \cdots r_{i_k}$ is a word in $W$, denote by $S_w$ the composition $S_{r_{i_1}} \circ \cdots \circ S_{r_{i_k}}$. We also continue the abuse of notation $S_w(C)$ where we mean the chamber whose vertices are determined by the action of $S_w$ on the vertices of $C$. Finally we can connect the notion of adjacency given in terms of the Coxeter group with that of the reflections described in terms of elementary divisors.

**Proposition 2.3.** *Let $C$ be the fundamental chamber in the apartment $\Sigma$ with vertices $[0,\dots,0]$, $[0,\dots,0,1]$, $\dots$ $[0,1,\dots,1]$, and associate to $C$ a group element $g \in W$ ($g \leftrightarrow C$). Then*

1. *For any element $v \in W$, the chamber $gv \leftrightarrow S_v(C)$.*
2. *For any chamber labelled by $h \in W$, let $C_h$ denote the corresponding maximal flag of lattices (i.e., $h \leftrightarrow C_h$). Then for any $w \in W$, the chamber $hw \leftrightarrow S_{vwv^{-1}}(C_h)$, where $h = gv$.*

*Proof.* We show $gv \leftrightarrow S_v(C)$ by induction on the length of $v$. When $v$ has length one, $v = r_i$ for some $i$, and by design $gr_i \leftrightarrow S_{r_i}(C)$. To complete the induction we need only show that $g(vr) \leftrightarrow S_{vr}(C)$ where $r$ is any generator $r_i \in W$. We have seen that $C$ and $S_r(C)$ are adjacent chambers, and by the base step of the induction, $gr \leftrightarrow S_r(C)$. Therefore since $S_v$ is a composition of reflections and hence a rigid motion, $S_v(C)$ and $S_v(S_r(C))$ are adjacent chambers. Since $g \leftrightarrow C$ and $gr \leftrightarrow S_r(C)$ are "$r$-adjacent", so are $S_v(C)$ and $S_v(S_r(C))$. This means that if $h \leftrightarrow S_v(S_r(C))$ and $\ell \leftrightarrow S_v(C)$, then $h = \ell r$. By induction $\ell \leftrightarrow S_v(C) \leftrightarrow gv$, hence $S_v(S_r(C)) \leftrightarrow h = \ell r = (gv)r = g(vr)$ as required.

For the second part of the proposition, suppose that $h \leftrightarrow C_h$. Write $h = gv$ for a unique $v \in W$. Then by the first part of the proposition, $C_h = S_v(C)$ and $hw = gvw \leftrightarrow S_{vw}(C)$. Observing $S_{vw}(C) = S_{vwv^{-1}}(S_v(C)) = S_{vwv^{-1}}(C_h)$ completes the proof. $\square$

Now that we have a good grasp on different characterizations of distance between chambers, we would like to define the distance between two vertices in the building. An obvious choice is to define the distance to be the minimal length among all galleries whose initial chamber contains one vertex and whose ending chamber contains the other. First we show that this is well-defined.

**Proposition 2.4.** *Let $q$ be the cardinality of the residue class field $k$ of $K_{\mathfrak{P}}$. Then for $n \geq 2$, every vertex in $\Delta_n$ is contained in precisely $r_n$ chambers where*

$$r_n = \frac{q^n-1}{q-1} \cdot \frac{q^n-q}{q(q-1)} \cdot \frac{q^n-q^2}{q^2(q-1)} \cdots \frac{q^n-q^{n-2}}{q^{n-2}(q-1)}.$$

*Remark 2.5.* For computational convenience, we note that $r_2 = q+1$ and $r_{n+1} = \left(\frac{q^{n+1}-1}{q-1}\right) r_n$.

*Proof.* A vertex is contained in a chamber (i.e., is a 0-simplex in the chamber) if and only if the chamber is part of the residue of the vertex. By [6, Chapter 9.2], such a residue is isomorphic to a building of type $A_{n-1}(k)$. By [6, Chapter 1.2], the chambers of this spherical building are maximal flags in an $n$-dimensional vector space $U$ over $k$. That is, the chambers are nested sequences of subspaces

$$0 \subset U_1 \subset U_2 \subset \cdots \subset U_{n-1} \subset U \text{ (with } \dim(U_i) = i).$$

To count the number of sequences is elementary: Any 1-dimensional space is spanned by a single nonzero vector of which there are $q^n - 1$ in $U$. For each of these vectors, any nonzero multiple of it generates the same subspace, and this is the only possible redundancy. Thus there are $(q^n-1)/(q-1)$ possible $U_1$. Suppose that $U_1 \subset \cdots \subset U_j$ have been chosen. To find all possible $U_{j+1} \supset U_j$, we need only ask how many distinct subspaces of dimension $j+1$ can be generated by adding a single vector to a basis of $U_j$. Similar to the first case, there are $q^n - q^j$ choices of vectors in $U$ not in $U_j$. To determine the redundancy, we observe that for $u, w \in U \setminus U_j$, $U_j \oplus \langle u \rangle = U_j \oplus \langle w \rangle$ if and only if $w = \lambda u + v$ for $v \in U_j$ and $\lambda \neq 0$. This yields $(q^n - q^j)/q^j(q-1)$ possible $U_{j+1}$. From here the proposition is clear. □

This generalizes the well-known $SL_2$ case in which the chambers are 1-simplices and the building $\Delta_2$ is a tree. The proposition states that the tree is $(q+1)$-regular.

## 3. Hecke Operators and the Bruhat-Tits Building

In this section we demonstrate that the number of vertices in the building $\Delta_n$ at distance one from a given vertex is the degree of a certain Hecke operator. In the case of $SL_2$, this provides another (cf. Proposition 2.4) proof of the $(q+1)$-regularity of the tree (building) $\Delta_2$. As suggested by the special cases $3 \leq n \leq 5$ in Proposition 3.5 below, for $n \geq 3$, it is unlikely that the degree of the Hecke operator is equal to the number given in Proposition 2.4, however there does appear to be an intriguing relationship between the two numbers.

We shall need generalizations of a few theorems of Shimura [8]. Let $V$ be our fixed $n$-dimensional vector space over $K_{\mathfrak{P}}$. For two lattices $L$ and $M$ on $V$, let $\{L : M\}$ denote the set of elementary divisors of $M$ in $L$. Fix a lattice $L$ on $V$ and a basis of $L$ which identifies $GL(L)$ with $GL_n(\mathcal{O}_{\mathfrak{P}})$. Then the group $G = GL_n(\mathcal{O}_{\mathfrak{P}})$ acts (on the left) on the set of lattices on $V$.

The proof of the following proposition is essentially identical to Shimura's Lemma 3.12 [8] with $SL_2(\mathbb{Z})$ replaced by $G$, since he uses only that the ring over which he is working is a PID.

**Proposition 3.1.** *Let $L$, $M$, $N$ be lattices on $V$. Then $\{L : M\} = \{L : N\}$ if and only if there exists a $g \in G$ such that $gM = N$.*

For elements $\alpha_1, \ldots, \alpha_n \in \mathcal{O}_{\mathfrak{P}}$, let $\langle \alpha_1, \ldots, \alpha_n \rangle$ denote the diagonal matrix in $M_n(\mathcal{O}_{\mathfrak{P}})$ with diagonal entries $\alpha_1, \ldots, \alpha_n$. Let $\xi = \langle \pi^{a_1}, \ldots, \pi^{a_n} \rangle \in M_n(\mathcal{O}_{\mathfrak{P}})$ with the $a_i \in \mathbb{Z}$, $a_1 \leq \ldots \leq a_n$. It is easily seen that $\xi \in \widetilde{G}$, the commensurator of $G$, so by Proposition 3.1 of [8] we have

$$G\xi G = \cup_{i=1}^{d} G\alpha_i = \cup_{j=1}^{e} \beta_j G$$

where $d = [G : G \cap \xi^{-1} G \xi]$ and $e = [G : G \cap \xi G \xi^{-1}]$ are finite. The integer $d$ is called the degree of the double coset $G\xi G$.

Let $\Delta(\ell) = \{M \in M_n(\mathcal{O}_{\mathfrak{P}}) \mid \mathrm{ord}_{\mathfrak{P}}(\det(M)) = \ell\}$. The existence of a Smith normal form tells us that for any $\alpha \in \Delta(\ell)$, $G\alpha G = G\langle \pi^{a_1}, \dots, \pi^{a_n}\rangle G$ where $0 \le a_1 \le \dots \le a_n$ are rational integers, and $\sum_1^n a_i = \ell$. By the class number formula [2, pp 19-20], for any element $\alpha \in \Delta(\ell)$, $G\alpha G = \cup_\nu G\alpha_\nu$ with finitely many cosets. In particular, $G\alpha G$ has finite degree. With a proof identical to Lemma 3.13 of [8], we have

**Proposition 3.2.** *Let $\xi = \langle \pi^{a_1}, \dots, \pi^{a_n}\rangle \in \Delta(\ell)$ as above, and let $L$ be a lattice on $V$. Then there is a 1–1 correspondence between cosets $G\xi_\nu$ in $G\xi G$ and lattices $M$ on $V$ such that $\{L : M\} = \{\pi^{a_1}, \dots, \pi^{a_n}\}$.*

We now turn to our task of determining how many vertices of our building are at distance 1 from a fixed vertex. It follows from Proposition 2.4 that the number of chambers containing a given vertex is finite, so it makes sense to define the distance between two vertices as the minimal length among all galleries whose initial chamber contains one vertex and whose ending chamber contains the other.

**Theorem 3.3.** *Let $v$ be any vertex in the building $\Delta_n$. Then the number of vertices in $\Delta_n$ which are distance one from $v$ is equal to*

1. *the degree of $G\langle 1, \pi\rangle G$ if $n = 2$, and*
2. *the degree of $G\langle 1, \pi, \dots, \pi, \pi^2\rangle G$ if $n \ge 3$.*

*Remark 3.4.* In a global setting, with $G = SL_2(\mathbb{Z})$ and $\pi = p$ a prime, the double coset $G\begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix} G$ is the standard Hecke operator $T_p$ which acts on modular forms of integral weight. This is our motivation for referring to these double cosets as Hecke operators. Serre defined related operators as formal sums on the vertices of $\Delta_2$ [7]. These are also called Hecke operators as they satisfy recursion relations paralleling those of the standard Hecke operators.

*Proof.* The differences between the two cases stem principally from the very different Coxeter diagrams which describe the buildings for $SL_2$ and for $SL_n$, $n \ge 3$. Since the result stated in this proposition is known for $n = 2$, we start the proof with the case of $n \ge 3$, and at the end of the proof comment about the differences which govern the $SL_2$ case.

Let $v$ and $w$ be vertices in the building $\Delta_n$. It is clear from the definition of distance given in terms of galleries that vertices $v$ and $w$ are distance one apart if and only if they are contained in chambers $C_v$ and $C_w$ which share a codimension-1 face, and where $C_w$ is not in the residue of $v$ (i.e. $v \notin C_w$).

As we saw above, $C_v$ and $C_w$ lie in a common apartment $\Sigma$. Let $L$ and $M$ be lattices on $V$ so that $v = [L]$ and $w = [M]$. By Remark 2.2, we may assume that relative to some basis $v = [0, \dots, 0]$ and $w = [0, a_2, \dots, a_n]$. The vertex $v$ is contained in the fundamental chamber $C$ of $\Sigma$ with vertices $[0, \dots, 0]$, $[0, \dots, 0, 1]$, $\dots$, $[0, 1, \dots, 1]$.

From the discussion preceding Proposition 2.3, we know the reflections $S_{r_i}$ each fix $[0, \dots, 0]$ if and only if $1 \le i \le n-1$, hence the residue of $v$ in $\Sigma$ consists of the chambers $S_u(C)$, where $u \in \langle r_i \mid 1 \leq i \leq n-1 \rangle$. Thus $C_v = S_\xi(C)$ for some $\xi \in \langle r_i \mid 1 \leq i \leq n-1 \rangle$. Let $g \in W$ correspond to the fundamental chamber $C$. Then by Proposition 2.3, $C_v \leftrightarrow g\xi$. Moreover, since $C_v$ and $C_w$ are adjacent, but $C_w$ is not in the residue of $C_v$, $C_w \leftrightarrow g\xi r_n$, that is $C_w = S_{\xi r_n}(C)$ by Proposition 2.3. This means that $w = S_{\xi r_n}(v) = S_{\xi r_n}([0, \dots, 0]) = S_\xi([0, 1, \dots, 1, 2])$. On the other hand, each $S_{r_i}$ for $i < n$ simply permutes the coordinates. Thus $w = [a_1, \dots, a_n]$, where $\{a_1, \dots, a_n\}$ is simply a permutation of $\{0, 1, \dots, 1, 2\}$. By permuting the original basis for $L$, we may assume that $v = [L] = [0, \dots, 0]$ and $w = [M] = [0, 1, \dots, 1, 2]$.

Thus to find all vertices distance one from $v$, we need only seek all classes of lattices $[M]$ for which $[M]$ has elementary divisors $[0, 1, \dots, 1, 2]$ relative to $[L]$. Since we are concerned only with classes of lattices, this is the same as finding all lattices $M$ so that with respect to some $\mathcal{O}_\mathfrak{P}$-basis of $L$, the elementary divisors of $M$ in $L$, $\{L : M\}$, are $\{1, \pi, \dots, \pi, \pi^2\}$. By Proposition 3.2, this number is the same as the degree of the double coset $G\langle 1, \pi, \dots, \pi, \pi^2 \rangle G$. Note that if $M_1$ and $M_2$ are distinct lattices which satisfy $\{L : M_1\} = \{L : M_2\}$, then $[M_1] \neq [M_2]$ and so correspond to distinct vertices, for if $M_1 = \lambda M_2$, $\{L : M_1\} = \{\lambda, \lambda\pi, \dots \lambda\pi, \lambda\pi^2\} = \{L : M_2\}$ if and only if $\lambda \in \mathcal{O}_\mathfrak{P}{}^\times$, and hence $M_1 = \lambda M_2 = M_2$.

For $n = 2$, there is a minor difference in the appearance of the final result arising from the difference of the Coxeter diagrams for $\widetilde{A}_1$ and $\widetilde{A}_m$ for $m \geq 2$. Using the distance function described by Serre [7] and discussed in the introduction, two classes, $[L]$ and $[M]$, of (rank 2) lattices are distance one apart if and only if $\{L : M\} = \{\pi^a, \pi^{a+1}\}$. Again since we are concerned only with classes of lattices, it suffices to find all lattices $M$ with $\{L : M\} = \{1, \pi\}$. Hence the number of vertices at distance one from a given vertex in the building for $SL_2$ is the degree of the double coset $G \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} G$, which is $q + 1$. As mentioned above, with $G = SL_2(\mathbb{Z})$ and $\pi = p$ a prime in $\mathbb{Z}$, this is the standard Hecke operator $T_p$ which acts on modular forms of integral weight. $\square$

We compute the degree of the Hecke operator in several cases.

**Proposition 3.5.** *Let $r_n$ be (as in Proposition 2.4) the number of chambers containing a given vertex in $\Delta_n$, and $\omega_n$ be the number of vertices having distance one from a given vertex in $\Delta_n$. Let $q$ denote the cardinality of the residue class field $\mathcal{O}_\mathfrak{P}/\pi\mathcal{O}_\mathfrak{P}$. Then*

1. $r_2 = \omega_2 = q + 1$,
2. $qr_3 = q(1 + 2q^1 + 2q^2 + q^3) = \omega_3$,
3. $qr_4 = (q + 1)\omega_4 = r_2\omega_4$, *and*
4. $qr_5 = (1 + 2q^1 + 2q^2 + q^3)\omega_5 = r_3\omega_5$.

*Remark 3.6.* Based solely upon the relatively complicated relations occuring above (and not on any insight into why this relationship exists), it is natural to conjecture that for $n \geq 3$, $qr_n = r_{n-2}\omega_n$.

*Proof.* The result is obtained by a straightforward, but relatively tedious computation. The result for $n = 2$ is well-known. We indicate the method for $n = 3$, and leave the remaining details to the reader. By consideration of the Smith normal form, it is immediate that

$$\Delta(3) = G\langle 1, 1, \pi^3\rangle G \cup G\langle 1, \pi, \pi^2\rangle G \cup G\langle \pi, \pi, \pi\rangle G.$$

Moreover, by Theorem II.4 of [2], the class number (i.e. the total number of right cosets in the union) is equal to $1+q+2q^2+2q^3+2q^4+q^5+q^6$ where $q$ is the cardinality of the residue class field $\mathcal{O}_{\mathfrak{P}}/\pi\mathcal{O}_{\mathfrak{P}}$. Obviously the 1 corresponds to the double coset $G\langle \pi, \pi, \pi\rangle G = G\langle \pi, \pi, \pi\rangle$, leaving the rest to distribute between the other two double cosets.

By Theorems II.2 and II.3 of [2], we may assume that every representative of the right cosets is in Hermite normal form. This done, we use Theorem II.10 of [2] to associate the right cosets to the appropriate double coset. In particular, we use that $G\alpha G = G\beta G$ if and only if $\alpha$ and $\beta$ have the same determinantal divisors. The result now follows from a case by case analysis. □

## 4. The case of $SL_3$

When $n = 3$, we know that the fundamental apartment is the Euclidean plane tiled by equilateral triangles. Recall that we may restrict our attention to galleries contained within the fundamental apartment $\Sigma$. Within the apartment, each vertex is contained in six equilateral triangles which together form a regular hexagon with the vertex at center. Moreover, each vertex is the center of a sequence of concentric hexagons (see Figure 4.1), expanding out from the vertex. This in turn gives a well defined notion of the *radial distance* from our vertex to any other vertex in the plane, the vertex at center having radial distance zero, the six vertices in the hexagon containing the vertex having radial distance 1, the next 12 vertices radially outward from the center having distance 2, and so on.

Because the fundamental apartment is a regular tiling of the plane, it is clear that a rigid motion will not alter distances, so there is no loss in considering the distance from an arbitrary vertex $[0, a, b]$ to the "origin" $[0, 0, 0]$. Consider three fundamental classes of lattices: $e_1 = [0, 1, 0]$, $e_2 = [0, 0, 1]$, and $e_3 = [0, 1, 1]$. We abuse the notation further by thinking of these classes of lattices as ordered triples in $\mathbb{Z}^3$. Then for any class of lattices $[0, a, b]$, we have:

$$[0, a, b] = ae_1 + be_2 = (a - b)e_1 + be_3 = (b - a)e_2 + ae_3.$$

**Proposition 4.1.** *The radial distance of a point* $[0, a, b]$ *from* $[0, 0, 0]$ *is given by*

$$r = \min\{\, |a| + |b|, |a - b| + |b|, |a - b| + |a| \,\} = \max\{\, |a|, |b|, |a - b| \,\}.$$

*Proof.* Since any two of the three vectors $e_i$ are linearly independent, the vertex $[0, a, b]$ can be expressed as a unique linear combination of any two of them. Consider the

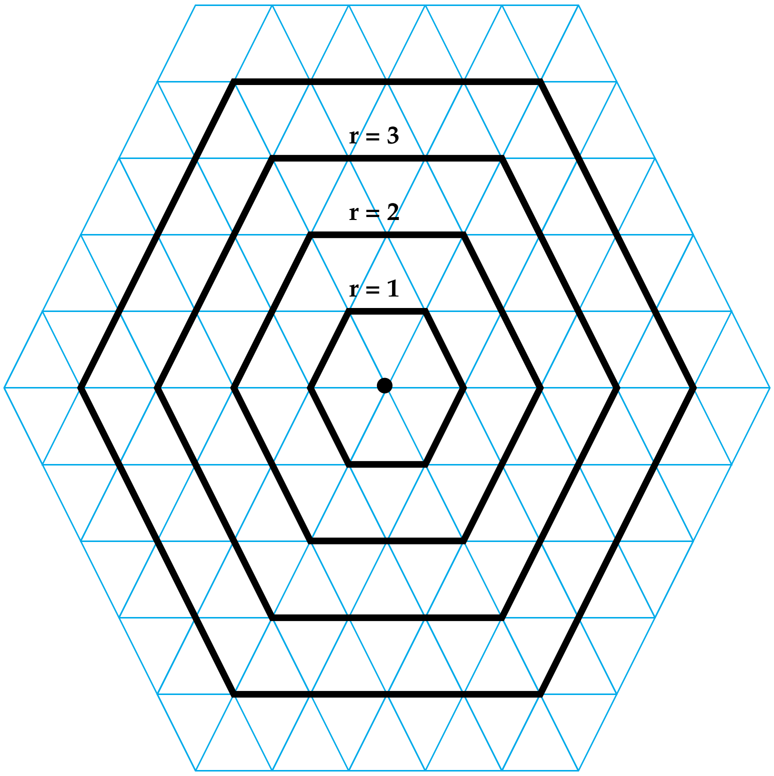


Figure 4.1

three sums of the absolute values of the coefficients corresponding to these linear combinations. The first equality of the proposition asserts that the radial distance is the smallest of these three sums.

Our three fundamental vectors $e_1$, $e_2$, and $e_3$ determine three lines through the origin in our fundamental apartment $\Sigma$. Clearly if $[0, a, b] = re_i$ for some $i$ then $r$ is the radial distance, and it is trivial to check that the coordinates satisfy the conditions of the proposition.

Otherwise $[0, a, b]$ lies in the interior of one of the six sectors determined by the lines containing the $e_i$. The point is then in the interior of the region bounded by two of the six rays $\varepsilon_i e_i$, for some $i = 1, 2, 3$, and $\varepsilon_i = \pm 1$ (see Figure 4.2); call them $f_1$ and $f_2$.

Since $f_1$ and $f_2$ are linearly independent, there are unique positive integers $s$ and $t$ for which $[0, a, b] = sf_1 + tf_2$. Moreover, it is clear that the radial distance $r = s + t$, since each move of unit length parallel to either vector $f_i$ crosses from one "radial band" to the next. That is, each such move of unit length proceeds from a vertex having radial distance $r$ to one having radial distance $r + 1$.

Let $f_3$ be whichever of the three vectors $e_i$ is not a multiple of $f_1$ or $f_2$. Note that vertices in the sector determined by the rays $f_1$ and $f_2$ which have a fixed

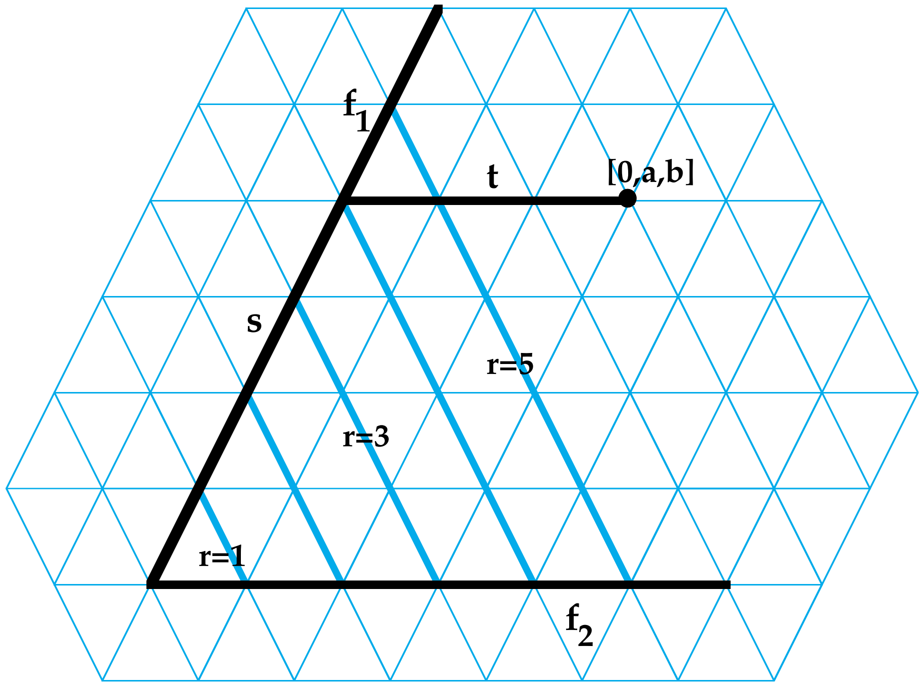


FIGURE 4.2

radial distance from $[0, 0, 0]$ lie on line segments which are parallel to $f_3$. Express $[0, a, b] = u_i f_i + v_i f_3$, $i = 1, 2$. We will show that $|u_i| + |v_i| \geq s + t$ which will establish the first part of the proposition by the comments in the first paragraph of the proof.

To establish $|u_i| + |v_i| \geq s + t$, we show that $|u_i| = s + t$. This is trivial since $u_i f_i$ brings us to a vertex having radial distance $|u_i|$ from $[0, 0, 0]$. Movement parallel to $f_3$ is through a set of vertices having a fixed radial distance from $[0, 0, 0]$, so the radial distance of $[0, a, b] = u_i f_i + v_i f_3$ is $|u_i| = s + t$.

Now a simple algebraic argument establishes that

$$\min\{\,|a| + |b|, |a - b| + |b|, |a - b| + |a|\,\} = \max\{\,|a|, |b|, |a - b|\,\}.$$

□

**Theorem 4.2.** *The distance from* $[0, 0, 0]$ *to* $v = [0, a, b]$ *is given by*

$$d = \begin{cases} 2(r-1) & \text{if } v = \lambda e_i, \lambda \in \mathbb{Z} \\ 2(r-2)+1 & \text{otherwise} \end{cases}$$

*where* $r$ *is the radial distance from* $[0, 0, 0]$ *to* $v$.

*Proof.* First we observe that $v$ has distance zero from $[0, 0, 0]$ if and only if $v = \pm e_i$ and $r = 1$ since such vertices are vertices of the chambers which form the residue of $[0, 0, 0]$ in the apartment $\Sigma$. This much agrees with the formula, so we assume $r \geq 2$.

We consider two cases, radial and nonradial, corresponding to whether or not $v = \lambda e_i$. Consider the radial case first. Then $r = |\lambda| \geq 2$, and the situation is pictured in Figure 4.3. It is clear that a minimal gallery must lie within the "strip" containing the residues (hexagons) of $[0,0,0]$ and $v$, for any gallery which leaves the strip must still cross at least as many walls (once outside the strip) and must also cross a wall to exit and one to reenter the strip. It is then clear from the figure that a minimal gallery has length $2(r-1)$ which completes the proof in the radial case.

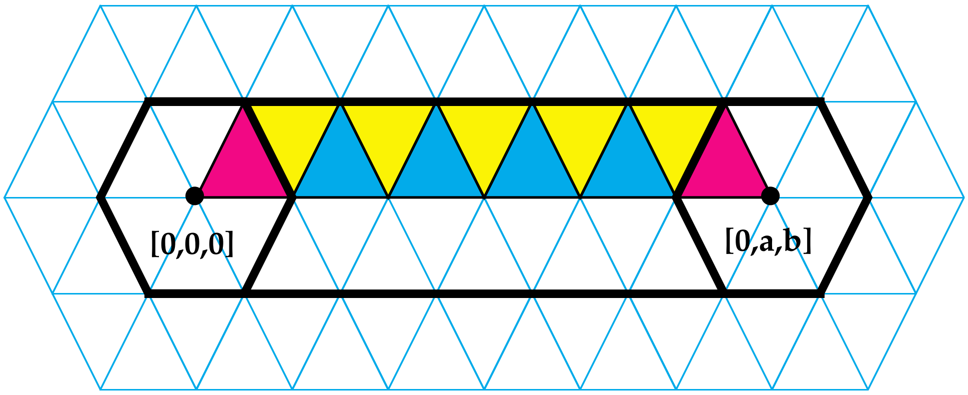


FIGURE 4.3

In the nonradial case, to proceed from $[0,0,0]$ to a vertex at radial distance 2 requires crossing at least one wall. To proceed from this vertex to any vertex of radial distance 3 requires crossing at least two more walls. It is then clear that it requires crossing at least $1+2(r-2)$ walls to proceed from $[0,0,0]$ to any vertex of radial distance $r$ ($r \geq 2$). This provides a lower bound for the distance. To establish an upper bound, we simply provide a gallery of length $1+2(r-2)$. Since $v$ is not a radial point, it lies in a unique sector. As in Proposition 4.1, we can write $v = [0,a,b] = sf_1 + tf_2$ with $s$ and $t$ positive integers. Then the radial distance $r = s+t$. It is clear from Figure 4.4 that the highlighted gallery is simply the combination of a radial gallery followed by a "radial-like" gallery whose lengths are $2(s-1)+2(t-1)+1$, where the "1" accounts for the crossing from the "s"-gallery to the "t"-gallery. We therefore have a gallery of length $2(s+t-2)+1 = 2(r-2)+1$ which provides our upper bound and completes the proof. $\square$

Next we return to the question of the number of vertices at distance one from a given vertex. A general answer to the problem was given in Theorem 3.3 in terms of the degree of a double coset. Given an explicit distance function for the $n=3$ case, we now give a brief alternate proof.

Fix a lattice $L$ on $V$ and let $M$ be any other lattice on $V$. Recall that the vertices of our building are in 1–1 correspondence with the classes of lattices on $V$, so we may feel free to scale lattices. If $\{L : M\} = \{\pi^{a_1}, \pi^{a_2}, \pi^{a_3}\}$ with $a_1 \leq a_2 \leq a_3$,

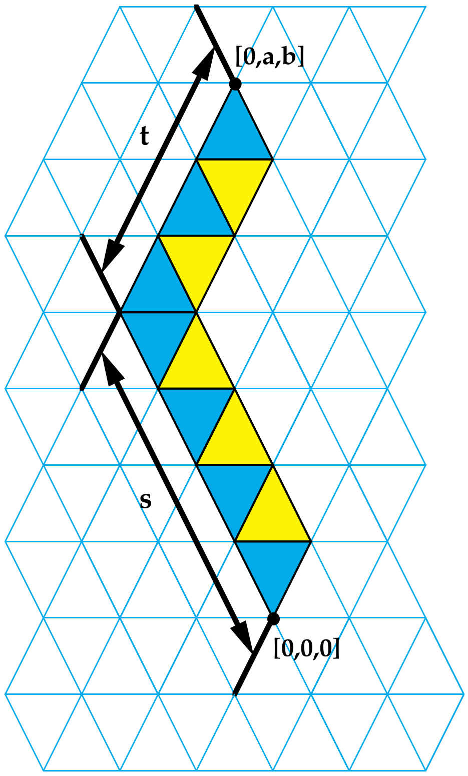


FIGURE 4.4

then $\{L : \pi^{-a_1}M\} = \{1, \pi^{a_2-a_1}, \pi^{a_3-a_1}\}$. In the notation of §2, $[L] = [0,0,0]$, and $[M] = [0,a,b]$, where $0 \leq a = a_2 - a_1 \leq b = a_3 - a_1$.

Applying Proposition 4.1 and Theorem 4.2, we see that the radial distance from $[M]$ to $[L]$ is $b$, and that the distance $d = 1$ if and only if $a = 1$ and $b = 2$. Thus we need to find all lattices with elementary divisors $\{L : M\} = \{\pi^a, \pi^{a+1}, \pi^{a+2}\}$. Since we are concerned only with the classes of the lattices, it suffices to consider all lattices with elementary divisors $\{L : M\} = \{1, \pi, \pi^2\}$. By Proposition 3.2, there is a 1–1 correspondence between such lattices and the right cosets $G\xi_\nu$ in $G\langle 1, \pi, \pi^2\rangle G$. Also note for any two such lattices $M_1$ and $M_2$, that $[M_1] \neq [M_2]$ since $M_2 = \lambda M_1$ implies $\{L : M_2\} = \{\lambda, \pi\lambda, \pi^2\lambda\} = \{1, \pi, \pi^2\}$ if and only if $\lambda$ is a unit in $\mathcal{O}_{\mathfrak{P}}$, in which case $M_1 = M_2$. We summarize this result as:

**Proposition 4.3.** *In the Bruhat-Tits building for $SL_3(K_{\mathfrak{P}})$ the number of vertices at distance one from a given vertex is equal to the number of of right cosets $G\xi_\nu$ in the double coset $G\langle 1, \pi, \pi^2\rangle G$.*

## 5. Class Numbers

Now we return to the global setting and consider the global central simple algebra $A$ over the number field $K$. Recall that we have chosen a finite set $S$ of primes of $K$, containing all the infinite ones, for which $R = \mathcal{O}^{(S)}$ has strict class number one (when $K = \mathbb{Q}$, we may assume that $R = \mathbb{Z}$). Actually, we require this assumption only to invoke Eichler's theorem in Theorem 5.3. We have fixed a maximal $R$-order $\Lambda$. Note that because $R$ is a PID, $\Lambda$ is a free $R$-module. As in §1, let $\mathfrak{P}$ be a finite prime of $K$ not contained in the set $S$. We also assume that $\mathfrak{P}$ is unramified in $A$. Let $R^{(\mathfrak{P})} = \mathcal{O}^{(S \cup \{\mathfrak{P}\})}$, and let $\Lambda^{(\mathfrak{P})} = \Lambda \otimes_R R^{(\mathfrak{P})}$. The goal of this section is to characterize the class number of $\Lambda$ as the number of orbits of the unit group ${\Lambda^{(\mathfrak{P})}}^\times$ acting on a global analog of our Bruhat-Tits building for $SL_n(K_\mathfrak{P})$.

Let $X$ consist of the set of maximal $R$-orders $\Gamma$ in $A$ such that $\Gamma_\mathfrak{q} = \Lambda_\mathfrak{q}$ for all primes $\mathfrak{q}$ of $R$, $\mathfrak{q} \neq \mathfrak{P}$. This is the same as the set of primes $\mathfrak{q}$ of $\mathcal{O}$ not in $S \cup \{\mathfrak{P}\}$. By the local-global correspondence for orders, the elements in $X$ are in 1–1 correspondence with the vertices of the Bruhat-Tits building $\Delta_n$ for $SL_n(K_\mathfrak{P})$. We make the set $X$ into a graph by defining the distance between two orders to be the distance between their completions at the prime $\mathfrak{P}$, and by placing an edge between two orders in $X$ if and only if they are distance one apart. Then from section 3 we know that $X$ is a regular graph. Note that $X$ is not necessarily a connected graph. For $n = 3$, the graph induced on an apartment is the union of three isomorphic connected components, each (combinatorially) isomorphic to the original apartment. We can actually say more about the structure of the graph $X$ utilizing the correspondence between $X$ and the building $\Delta_n$; however, we do not require the knowledge of any additional structure for our current purposes.

Let $J_A$ denote the ideles of the algebra $A$. As a set

$$J_A = \{\widetilde{\alpha} = (\alpha_\mathfrak{p}) \in \prod_\mathfrak{p} A_\mathfrak{p}^\times \mid \alpha_\mathfrak{p} \in \Lambda_\mathfrak{p}^\times \text{ for almost all } \mathfrak{p}\}$$

where the product is over all primes $\mathfrak{p}$ of $K$ including the infinite ones.

Since all maximal orders in the local algebras are conjugate (in fact, unique when $A_\mathfrak{p}$ is a division algebra), the local-global correspondence between orders tells us that the ideles $J_A$ act transitively on the elements of $X$:

$$\Lambda \leftrightarrow \{\Lambda_\mathfrak{p}\} \leftrightarrow \{\alpha_\mathfrak{p}^{-1}\Lambda_\mathfrak{p}\alpha_\mathfrak{p}\} = \{\Gamma_\mathfrak{p}\} \leftrightarrow \Gamma = \widetilde{\alpha}^{-1}\Lambda\widetilde{\alpha}.$$

By a left $\Lambda$-ideal in $A$ we mean an $R$-lattice $\mathcal{I}$ on $A$ such that $\mathcal{I}_\mathfrak{p} = \Lambda_\mathfrak{p}\alpha_\mathfrak{p}$ for all finite $\mathfrak{p} \in R$ and $\alpha_\mathfrak{p} \in A_\mathfrak{p}^\times$. Using the Invariant Factor Theorem (Theorem 81.11 of [3]), it is easy to see that $\alpha_\mathfrak{p} \in \Lambda_\mathfrak{p}^\times$ for almost all primes $\mathfrak{p}$, so as shorthand, we will write $\mathcal{I} = \Lambda\widetilde{\alpha}$ where $\widetilde{\alpha} = (\alpha_\mathfrak{p}) \in J_A$ is any idele giving the proper localizations at the finite places. Two left $\Lambda$-ideals $\mathcal{I}$ and $\mathcal{J}$ are in the same ideal class if $\mathcal{I} = \mathcal{J}\alpha$ for some $\alpha \in A^\times$. Let

$$\mathcal{U}(\Lambda) = \{\widetilde{\alpha} = (\alpha_\mathfrak{p}) \in J_A \mid \alpha_\mathfrak{p} \in \Lambda_\mathfrak{p}^\times \text{ for all finite primes } \mathfrak{p} \text{ of } R\}.$$

Then the left $\Lambda$-ideals are in 1–1 correspondence with the cosets $\mathcal{U}(\Lambda)\backslash J_A$, and the left $\Lambda$-ideal classes are in 1–1 correspondence with the double cosets $\mathcal{U}(\Lambda)\backslash J_A/A^\times$.

The number of such ideal classes is finite (e.g., see Theorem 26.4 of [4]), and is called the class number of $\Lambda$. We have now defined both objects we need to compare.

Consider the $R^{(\mathfrak{P})}$-order

$$\Lambda^{(\mathfrak{P})} = \Lambda \otimes_R R^{(\mathfrak{P})} = \cap_{\mathfrak{q}\neq\mathfrak{P}}(\Lambda_{\mathfrak{q}} \cap A)$$

where the intersection is over all finite $R$-primes $\mathfrak{q}$. Note that every maximal order $\Gamma$ in $X$ generates the same $R^{(\mathfrak{P})}$-order since $\Lambda_{\mathfrak{q}} = \Gamma_{\mathfrak{q}}$ for all primes $\mathfrak{q} \neq \mathfrak{P}$ of $R$. Let $G = {\Lambda^{(\mathfrak{P})}}^{\times}$, the unit group of $\Lambda^{(\mathfrak{P})}$.

**Proposition 5.1.** *There is a natural group action of the group $G$ on the graph $X$ in which $G$ acts as a group of isometries.*

*Proof.* As usual, embed $A^{\times}$ into $J_A$ as the diagonal. Note that for all finite primes $\mathfrak{q} \neq \mathfrak{P}$ of $K$, $\Lambda_q^{(\mathfrak{P})} = \Lambda_{\mathfrak{q}}$ (and $\Lambda_{\mathfrak{q}} = A_{\mathfrak{q}}$ for $\mathfrak{q}$ a finite prime in $S$). Consequently, $\alpha \in {\Lambda^{(\mathfrak{P})}}^{\times}$ implies that $\alpha \in \Lambda_{\mathfrak{q}}^{\times}$ for all $\mathfrak{q} \neq \mathfrak{P}$. Then $G = {\Lambda^{(\mathfrak{P})}}^{\times}$ acts by conjugation on the maximal orders in $X$: For $\Gamma \in X$, and $\alpha \in G$, we have

$$\alpha^{-1}\Gamma\,\alpha \leftrightarrow \{\alpha^{-1}\Gamma_{\mathfrak{q}}\,\alpha\} \leftrightarrow \begin{Bmatrix} \Gamma_{\mathfrak{q}} = \Lambda_{\mathfrak{q}} & \text{if } \mathfrak{q} \neq \mathfrak{P} \\ \alpha^{-1}\Gamma_{\mathfrak{P}}\,\alpha & \text{if } \mathfrak{q} = \mathfrak{P} \end{Bmatrix} \leftrightarrow \begin{Bmatrix} \Lambda_{\mathfrak{q}} & \text{if } \mathfrak{q} \neq \mathfrak{P} \\ \alpha^{-1}\Gamma_{\mathfrak{P}}\,\alpha & \text{if } \mathfrak{q} = \mathfrak{P} \end{Bmatrix} \in X,$$

so $G$ acts on the vertices of $X$. To show that the action is an isometry and in particular to show the action takes edges to edges, choose two vertices $\Gamma, \Gamma' \in X$. By definition, the distance between two orders in $X$ is defined to be the distance between their completions at $\mathfrak{P}$ in $\Delta_n$. Thus if $\alpha \in G$, we need only show that the distance between $\Gamma_{\mathfrak{P}}$ and $\Gamma'_{\mathfrak{P}}$ is the same as the distance between $\alpha^{-1}\Gamma_{\mathfrak{P}}\,\alpha$ and $\alpha^{-1}\Gamma'_{\mathfrak{P}}\,\alpha$. Recall that all maximal orders in $A_{\mathfrak{P}}$ are conjugate to $\Lambda_{\mathfrak{P}}$, so we set $\Gamma_{\mathfrak{P}} = \gamma\Lambda_{\mathfrak{P}}\gamma^{-1}$ and $\Gamma'_{\mathfrak{P}} = \gamma'\Lambda_{\mathfrak{P}}\gamma'^{-1}$, $\gamma, \gamma' \in A_{\mathfrak{P}}^{\times}$. Let $L$ be a lattice on the vector space $V$ (over $K_{\mathfrak{P}}$) so that $\Lambda_{\mathfrak{P}} = \mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(L)$. Then $\Gamma_{\mathfrak{P}} = \mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(\gamma L)$ and $\Gamma'_{\mathfrak{P}} = \mathrm{End}_{\mathcal{O}_{\mathfrak{P}}}(\gamma' L)$. From Remark 2.2, we see that the distance between orders is completely determined by the elementary divisors of the lattices which determine the orders. Thus the distance between $\Gamma_{\mathfrak{P}}$ and $\Gamma'_{\mathfrak{P}}$ is completely determined by the elementary divisors $\{\gamma L : \gamma' L\}$ while the distance between $\alpha^{-1}\Gamma_{\mathfrak{P}}\,\alpha$ and $\alpha^{-1}\Gamma'_{\mathfrak{P}}\,\alpha$ is completely determined by the elementary divisors $\{\alpha^{-1}\gamma L : \alpha^{-1}\gamma' L\}$. Using the linearity of $\alpha$, it is trivial to see that these sets of elementary divisors are equal; hence our group action determines an isometry. □

Before stating our main result, it is convenient to remind the reader of Eichler's norm theorem (e.g. see Theorem 34.9 of [4]).

**Theorem 5.2** (Eichler)**.** *Let $R$ be a Dedekind domain with quotient field $K$, and let $A$ be a central simple algebra over $K$ which satisfies the Eichler condition over $R$. Let $\mathcal{L}$ be a normal ideal in $A$. Then $\mathcal{L}$ is a principal ideal if and only if its reduced norm is a principal ideal $R\alpha$ for $\alpha \in \mathfrak{u}(A)$.*

Here, the Eichler condition is always satisfied for central simple algebras of dimension $n^2$, $n \geq 3$ (see [4, Remark 34.4]). Also, a normal ideal is one whose left- (or equivalently right-) order is maximal, and

$$\mathfrak{u}(A) = \{\alpha \in K^\times \mid \alpha_\mathfrak{p} > 0 \text{ for all infinite } \mathfrak{p} \text{ for which } A_\mathfrak{p} \text{ is ramified}\}.$$

Note that because we constructed $R$ to have strict class number one, the norm of any normal ideal in $A$ is a principal ideal with totally positive generator, and hence automatically satisfies the requirements of Eichler's theorem.

**Theorem 5.3.** *The number of orbits in $X$ under the action of $G = \Lambda^{(\mathfrak{P})^\times}$ is finite and equal to the class number of $\Lambda$.*

*Proof.* We prove our result by establishing a bijection between the following sets:

$$\mathcal{U}(\Lambda)\backslash J_A/A^\times \leftrightarrow \Lambda_\mathfrak{P}^\times K_\mathfrak{P}^\times \backslash A_\mathfrak{P}^\times/\Lambda^{(\mathfrak{P})^\times} \leftrightarrow X/G.$$

Since $\Lambda$ is a maximal $R$-ideal, $\Lambda^{(\mathfrak{P})}$ is a maximal $R^{(\mathfrak{P})}$-ideal (see e.g., Chapter IV of [5]). By Eichler's theorem and our condition on $R$, we see that $\Lambda^{(\mathfrak{P})}$ has class number one, that is,

$$|\mathcal{U}(\Lambda^{(\mathfrak{P})})\backslash J_A/A^\times| = 1 \qquad \text{or} \qquad J_A = \mathcal{U}(\Lambda^{(\mathfrak{P})})A^\times.$$

Also, since $R$ has class number one, we have that the ideles of $K$, $K_\mathbf{A}^\times$, can be expressed $K_\mathbf{A}^\times = \mathcal{U}(R)K^\times$. For notational convenience, we "decompose" the groups $\mathcal{U}(\Lambda^{(\mathfrak{P})})$ and $\mathcal{U}(R)$ and write

$$J_A = A_\infty^\times \cdot A_\mathfrak{P}^\times \cdot \prod_{\mathfrak{q}\neq\mathfrak{P}} \Lambda_q^{(\mathfrak{P})^\times} \cdot A^\times \qquad \text{and} \qquad K_\mathbf{A}^\times = K_\infty^\times \cdot R_\mathfrak{P}^\times \cdot \prod_{\mathfrak{q}\neq\mathfrak{P}} R_\mathfrak{q}^\times \cdot K^\times.$$

where $A_\infty^\times$ and $K_\infty^\times$ denote (respectively) the product of $A_\mathfrak{p}^\times$ (respectively $K_\mathfrak{p}^\times$) over the infinite primes of $K$. We recall that for finite primes $\mathfrak{q} \neq \mathfrak{P}$ we have $\Lambda_q^{(\mathfrak{P})} = \Lambda_\mathfrak{q}$. For $\mathfrak{q} \in S$ (finite), $\Lambda_q^{(\mathfrak{P})} = \Lambda_\mathfrak{q} = A_\mathfrak{q}$ and $R_\mathfrak{q} = K_\mathfrak{q}$, while for $\mathfrak{q} \notin S$ ($\mathfrak{q} \neq \mathfrak{P}$), $\Lambda_q^{(\mathfrak{P})} = \Lambda_\mathfrak{q} \neq A_\mathfrak{q}$ and $R_\mathfrak{q} = \mathcal{O}_\mathfrak{q}$. In particular, for all finite primes $\mathfrak{q} \neq \mathfrak{P}$, $R_\mathfrak{q}^\times \subset \Lambda_\mathfrak{q}^\times$. Finally, $R_\mathfrak{P} = \mathcal{O}_\mathfrak{P}$. Thus,

$$J_A = A_\infty^\times \cdot A_\mathfrak{P}^\times \cdot \prod_{\mathfrak{q}\neq\mathfrak{P}} \Lambda_\mathfrak{q}^\times \cdot A^\times \qquad \text{and} \qquad K_\mathbf{A}^\times = K_\infty^\times \cdot \mathcal{O}_\mathfrak{P}^\times \cdot \prod_{\mathfrak{q}\neq\mathfrak{P}} R_\mathfrak{q}^\times \cdot K^\times.$$

Consistent with the above decompositions, write an element $\widetilde{\alpha} \in J_A$ as $\widetilde{\alpha} = (\alpha_\infty; \alpha_\mathfrak{P}, \alpha_0) \cdot a$; similarly, for $\widetilde{k} \in K_\mathbf{A}^\times$, write $\widetilde{k} = (k_\infty; k_\mathfrak{P}, k_0) \cdot k$. Consider the map

$$\Lambda_\mathfrak{P}^\times K_\mathfrak{P}^\times \backslash A_\mathfrak{P}^\times/\Lambda^{(\mathfrak{P})^\times} \to \mathcal{U}(\Lambda)\backslash J_A/A^\times \text{ defined by } \Lambda_\mathfrak{P}^\times K_\mathfrak{P}^\times a_\mathfrak{P}\Lambda^{(\mathfrak{P})^\times} \mapsto \mathcal{U}(\Lambda)(1; a_\mathfrak{P}, 1)A^\times.$$

First we show that this map is well-defined. We only need to show that

$$\mathcal{U}(\Lambda)(1; \lambda_{\mathfrak{P}} k_{\mathfrak{P}} a_{\mathfrak{P}} \lambda, 1)A^{\times} = \mathcal{U}(\Lambda)(1; a_{\mathfrak{P}}, 1)A^{\times}$$

where $\lambda_{\mathfrak{P}} \in \Lambda_{\mathfrak{P}}^{\times}$, $k_{\mathfrak{P}} \in K_{\mathfrak{P}}^{\times}$ and $\lambda \in \Lambda^{(\mathfrak{P})^{\times}}$. Using $K_{\mathbf{A}}^{\times} = \mathcal{U}(R)K^{\times}$, we see that $\mathcal{U}(\Lambda)K_{\mathbf{A}}^{\times}K^{\times} = \mathcal{U}(\Lambda)K^{\times}$. This, together with the fact that $K^{\times}$ is in the center of $A^{\times}$ and $A_{\mathfrak{q}}^{\times}$ for all primes $\mathfrak{q}$, yields $\mathcal{U}(\Lambda)(1; \lambda_{\mathfrak{P}} k_{\mathfrak{P}} a_{\mathfrak{P}} \lambda, 1)A^{\times} = \mathcal{U}(\Lambda)(1; a_{\mathfrak{P}}\lambda, 1)A^{\times}$. Now $\mathcal{U}(\Lambda)(1; a_{\mathfrak{P}}\lambda, 1)A^{\times} = \mathcal{U}(\Lambda)(\lambda^{-1}; a_{\mathfrak{P}}, \lambda^{-1})A^{\times}$ since $\Lambda^{(\mathfrak{P})^{\times}} \subset A^{\times}$ is embedded in $J_A$ as the diagonal. Since $\Lambda^{(\mathfrak{P})^{\times}}$ is contained in $A_{\infty}^{\times}$ and $\Lambda_{\mathfrak{q}}^{\times}$ for $\mathfrak{q} \neq \mathfrak{P}$, $(\lambda^{-1}, 1, \lambda^{-1}) \in \mathcal{U}(\Lambda)$ which yields the result.

Next we show that the map is onto: By Eichler's theorem, we have that $J_A = \mathcal{U}(\Lambda^{(\mathfrak{P})})A^{\times}$, and since $\mathcal{U}(\Lambda)$ and $\mathcal{U}(\Lambda^{(\mathfrak{P})})$ differ only in the $\mathfrak{P}$-th component, it is trivial to see that for any $\widetilde{\alpha} = (\alpha_{\mathfrak{p}}) \in J_A$, $\mathcal{U}(\Lambda)\widetilde{\alpha}A^{\times} = \mathcal{U}(\Lambda)(1; \alpha_{\mathfrak{P}}, 1)A^{\times}$.

Finally we show that the map is one-to-one: Suppose that

$$\mathcal{U}(\Lambda)(1; \alpha_{\mathfrak{P}}, 1)A^{\times} = \mathcal{U}(\Lambda)(1; \beta_{\mathfrak{P}}, 1)A^{\times}.$$

Then $(1; \alpha_{\mathfrak{P}}, 1) = (\lambda_{\infty}a, \lambda_{\mathfrak{P}}\beta_{\mathfrak{P}}a, \lambda_{\mathfrak{q}}a)$ where $(\lambda_{\infty}; \lambda_{\mathfrak{P}}, \lambda_{\mathfrak{q}}) \in \mathcal{U}(\Lambda)$ and $a \in A^{\times}$. It follows that $\alpha_{\mathfrak{P}} = \lambda_{\mathfrak{P}}\beta_{\mathfrak{P}}a$ where $\lambda_{\mathfrak{P}} \in \Lambda_{\mathfrak{P}}^{\times}$, and $a \in A^{\times}$, and $\lambda_{\mathfrak{q}}a = 1$ for all $\mathfrak{q} \neq \mathfrak{P}$. Thus $a \in \cap_{\mathfrak{q}\neq\mathfrak{P}}(\Lambda_{\mathfrak{q}}^{\times} \cap A^{\times}) = \Lambda^{(\mathfrak{P})^{\times}}$, and so $\alpha_{\mathfrak{P}} \in \Lambda_{\mathfrak{P}}^{\times}K_{\mathfrak{P}}^{\times}\beta_{\mathfrak{P}}\Lambda^{(\mathfrak{P})^{\times}}$ which completes the proof of injectivity.

Now we show that there is a bijection between $\Lambda_{\mathfrak{P}}^{\times}K_{\mathfrak{P}}^{\times}\backslash A_{\mathfrak{P}}^{\times}/\Lambda^{(\mathfrak{P})^{\times}}$ and the orbits $X/G$ of $G$ in $X$. Recall that the points in $X$ are maximal $R$-orders $\Gamma$ in $A$ satisfying $\Gamma_{\mathfrak{q}} = \Lambda_{\mathfrak{q}}$ for all $\mathfrak{q} \neq \mathfrak{P}$; thus the points in $X$ are in 1–1 correspondence with maximal $\mathcal{O}_{\mathfrak{P}}$-orders in $A_{\mathfrak{P}}$, all of which are of the form $\alpha_{\mathfrak{P}}^{-1}\Lambda_{\mathfrak{P}}\alpha_{\mathfrak{P}}$ for some $\alpha_{\mathfrak{P}} \in A_{\mathfrak{P}}^{\times}$. Now $\mathfrak{P}$ is unramified in $A$ by choice, and we have chosen to identify $A_{\mathfrak{P}}$ with $M_n(K_{\mathfrak{P}})$ in such a way that $\Lambda_{\mathfrak{P}} = M_n(\mathcal{O}_{\mathfrak{P}}) = M_n(R_{\mathfrak{P}})$. By Corollary 37.26 of [4], the normalizer of $\Lambda_{\mathfrak{P}}$ is $\Lambda_{\mathfrak{P}}^{\times} \cdot K_{\mathfrak{P}}^{\times}$, so the points of $X$ are in 1–1 correspondence with the cosets $\Lambda_{\mathfrak{P}}^{\times}K_{\mathfrak{P}}^{\times}\backslash A_{\mathfrak{P}}^{\times}$. Let $\Gamma \in X$, with $\Gamma_{\mathfrak{P}} = \alpha_{\mathfrak{P}}^{-1}\Lambda_{\mathfrak{P}}\alpha_{\mathfrak{P}}$ for some $\alpha_{\mathfrak{P}} \in K_{\mathfrak{P}}^{\times}$. An orbit of $\Gamma$ under the action of $G = \Lambda^{(\mathfrak{P})^{\times}}$ thus corresponds to the double coset $\Lambda_{\mathfrak{P}}^{\times}K_{\mathfrak{P}}^{\times}\alpha_{\mathfrak{P}}\Lambda^{(\mathfrak{P})^{\times}}$. This correspondence is obviously onto. Moreover, if $\Lambda_{\mathfrak{P}}^{\times}K_{\mathfrak{P}}^{\times}\alpha_{\mathfrak{P}}\Lambda^{(\mathfrak{P})^{\times}} = \Lambda_{\mathfrak{P}}^{\times}K_{\mathfrak{P}}^{\times}\beta_{\mathfrak{P}}\Lambda^{(\mathfrak{P})^{\times}}$, then $\beta_{\mathfrak{P}} = \lambda_{\mathfrak{P}}k_{\mathfrak{P}}\alpha_{\mathfrak{P}}\lambda$ for some $\lambda_{\mathfrak{P}} \in \Lambda_{\mathfrak{P}}^{\times}$, $k_{\mathfrak{P}} \in K_{\mathfrak{P}}^{\times}$ and $\lambda \in \Lambda^{(\mathfrak{P})^{\times}}$. Thus $\beta_{\mathfrak{P}}^{-1}\Lambda_{\mathfrak{P}}\beta_{\mathfrak{P}} = \lambda^{-1}\alpha_{\mathfrak{P}}^{-1}\Lambda_{\mathfrak{P}}\alpha_{\mathfrak{P}}\lambda$ (since $\lambda_{\mathfrak{P}}k_{\mathfrak{P}}$ is in the normalizer of $\Lambda_{\mathfrak{P}}$) is in the same orbit as $\alpha_{\mathfrak{P}}^{-1}\Lambda_{\mathfrak{P}}\alpha_{\mathfrak{P}}$. Hence the correspondence is one-to-one. □

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