Hecke Operators for the Symplectic Group and Buildings

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Abstract

We demonstrate that (isomorphic images of) the generators of the Hecke algebra for the symplectic group have generating series which are equal to highly structured rational functions. For two of the $(n+1)$ generators of the algebra, the denominators of those rational functions correspond to the standard and spinor zeta functions; in the remaining cases, they represent new zeta functions. We also give two representations of the classical Hecke algebra. The first of these is achieved by defining Hecke operators which act on the vertices of the Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$, while the second is related to but distinct from Satake's isomorphism between the local Hecke algebra and a ring of polynomials invariant under a certain Weyl group.

1 Introduction

Hecke theory for automorphic forms on the symplectic group is still very much in its infancy. Simplisticly, the major stumbling block is that unlike the elliptic modular case, there is no obvious connection between the known invariants of the Hecke algebra (Satake p-parameters) and the Fourier coefficients of a Hecke eigenform, although there has been some interesting work done: using a partial knowledge of Satake parameters to infer complete knowledge ([7]), or correlations between Fourier coefficients and Hecke eigenvalues in degree 2 ([4]). Still we are very far away from a satisfactory general theory.

In this paper, we have two major goals. The first is to demonstrate that (isomorphic images of) the generators of the Hecke algebra for the symplectic group have generating series which are equal to highly structured rational functions. For two of the $(n+1)$ generators of the algebra, the denominators of those rational functions correspond to the standard and spinor zeta functions; in the remaining cases, they represent new zeta functions. The second

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goal is to give two representations of the Hecke algebra. The first of these is achieved by defining Hecke operators which act on the vertices of the Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$, while the second is related to but distinct from Satake's isomorphism between the local Hecke algebra and a ring of polynomials invariant under a certain Weyl group. As a byproduct of the representation on the vertices of the building, we derive a number of what might be called "arithmetic" results about the building for Sp_n.

It is well-known (see e.g., Cartier [2], Theorem 4.1) that the Satake map shows that the p-part of the Hecke algebra associated to the symplectic group is isomorphic to a polynomial ring invariant under a certain Weyl group. In [1], Andrianov and Zhuravlev refer to this isomorphism as the spherical map, and give a rather complicated description of it in terms of right cosets of the double cosets which generate the Hecke algebra. In this paper we define a slightly different isomorphism with several advantages: the isomorphism is defined naturally in terms of the double cosets which generate the Hecke algebra (not in terms of right cosets), and derives its shape from a representation of the local Hecke algebra on the vertices of the Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$. This correspondence makes role of the Weyl group completely explicit. Then we exploit the isomorphism of the Hecke algebra with the polynomial ring by using that setting to show the images of the standard generators of the Hecke algebra have generating series which produce structured rational functions. Finally, we show that our representations are related by establishing the validity of a commutative diagram.

The key to understanding both representations is to understand the structure of the Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$, in particular how the vertices of an apartment are associated with symplectic elementary divisors relative to a fixed a lattice. It is of course here that the Weyl group plays a natural role. Moreover, it is through the natural connection between right cosets of a double coset and (sub)lattices of a given lattice with prescribed elementary divisors that we begin to develop our representations. In section 2, we describe the connection between lattices and right cosets. In section 3, we look carefully at the building for Sp_n, give labellings of an apartment in terms of symplectic divisors, and discuss how special vertices are arithmetically distinguished. We then begin to consider polynomials invariant under the Weyl group and a natural correspondence between vertices and monomials which forms the basis of our representations. Finally, we define our "Hecke" operators in the polynomial algebra. In the final section, we define the two representations and prove that they are indeed ring homomorphisms, and are related via a commutative diagram. At the end, we make a brief comparison of the Satake isomorphism and our map.

2 Preliminaries

2.1 The Classical Hecke Algebra

Much of this material can be found in Chapter 3 of [1]; we state it here to set the notation. Let $\Gamma = \Gamma_n = Sp_n(\mathbb{Z}) \subset SL_{2n}(\mathbb{Z})$, and let $G = GSp_n^+(\mathbb{Q}) \subset GL_{2n}(\mathbb{Q})$ be the group of symplectic similitudes with scalar factor $r(M) \in \mathbb{Q}_+^\times$:

$$\begin{aligned} GSp_n^+(\mathbb{Q}) &= \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n}(\mathbb{Q}) \mid A^tC = C^tA, B^tD = D^tB, A^tD - C^tB = r(M)I_{2n}\} \\ &= \{M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in M_{2n}(\mathbb{Q}) \mid AB^t = B^tA, CD^t = DC^t, AD^t - BC^t = r(M)I_{2n}\}. \end{aligned}$$

Let H denote the rational Hecke algebra associated to the pair Γ and G. That is, as a vector space, H is generated by all double cosets $\Gamma\xi\Gamma$ ($\xi \in G$), and we turn H into an algebra by defining the multiplication law as follows: Given $\xi_1, \xi_2 \in G$, define

$$\Gamma\xi_1\Gamma \cdot \Gamma\xi_2\Gamma = \sum_{\xi} c(\xi)\Gamma\xi\Gamma, \tag{2.1}$$

where the sum is over all double cosets $\Gamma\xi\Gamma \subseteq \Gamma\xi_1\Gamma\xi_2\Gamma$, and the $c(\xi)$ are nonnegative integers (see [6]). There is an alternate characterization of the Hecke algebra which will be convenient as well. Let $L(\Gamma, G)$ be the rational vector space with basis consisting of right cosets $\Gamma\xi$ for $\xi \in G$. The Hecke algebra can be thought of as those elements of $L(\Gamma, G)$ which are right invariant under the action of Γ. Thus we can and will think of a double coset $\Gamma\xi\Gamma = \cup\Gamma\xi_\nu$ as the disjoint union of right cosets and as the sum of those right cosets $\sum \Gamma\xi_\nu \in L(\Gamma, G)$.

The global Hecke algebra, H, is generated by local Hecke algebras, H_p, one for each prime p, obtained as above by replacing G by $G \cap GL_{2n}(\mathbb{Z}[p^{-1}])$ in the above construction. H_p is generated by double cosets $\Gamma\xi\Gamma$ with ξ of the form $\mathrm{diag}(p^{a_1}, \dots, p^{a_n}, p^{b_1}, \dots, p^{b_n})$ where $a_1 \le \dots \le a_n \le b_n \le \dots \le b_1$ are integers with $p^{a_i+b_i} = r(\xi)$ for all i. It is occasionally useful to consider the "integral" Hecke algebra $\underline{H}_p$ generated by all ξ as above with $\xi = \mathrm{diag}(p^{a_1}, \dots, p^{a_n}, p^{b_1}, \dots, p^{b_n}) \in M_{2n}(\mathbb{Z})$.

In this paper, we want to work with a p-adic version of the local Hecke algebra. This local Hecke algebra is isomorphic to a subalgebra of the Hecke algebra associated to the Hecke pair $\Gamma_p = Sp_n(\mathbb{Z}_p)$ and $G_p = GSp_n(\mathbb{Q}_p)$. To be explicit we set $\mathcal{H}_p$ to be the Hecke algebra generated as a $\mathbb{Q}$-vector space by double cosets $\Gamma_p\xi\Gamma_p$ with $\xi = \mathrm{diag}(p^{a_1}, \dots, p^{a_n}, p^{b_1}, \dots, p^{b_n})$ as above, and let $\underline{\mathcal{H}}_p$ denote the integral subring as above.

The key to observing the above isomorphism is the following observation:

Lemma 2.1. *Let $\xi \in GSp_n^+(\mathbb{Q}) \cap GL_{2n}(\mathbb{Z}[p^{-1}])$. If $\Gamma\xi\Gamma$ is the disjoint union $\cup\Gamma\alpha_i$, then $\Gamma_p\xi\Gamma_p$ is the disjoint union $\cup\Gamma_p\alpha_i$.*

Proof. Without loss of generality, we may assume $\xi \in M_{2n}(\mathbb{Z})$ since the general case follows by multiplying by a power of p. First, it is clear that $\cup\Gamma_p\alpha_i \subseteq \Gamma_p\xi\Gamma_p$, and that the α_i have integer entries. Next, we see that the cosets $\Gamma_p\alpha_i$ are disjoint since if not then $\alpha_i\alpha_j^{-1} \in \Gamma_p \cap GL_{2n}(\mathbb{Z}[p^{-1}]) \subseteq \Gamma$. To show that the union is all of $\Gamma_p\xi\Gamma_p$, we need only show that $\xi\Gamma_p \subseteq \Gamma_p\xi\Gamma$, for then any element $\tilde{\gamma}_1\xi\tilde{\gamma}_2 = \tilde{\gamma}_3\xi\gamma$ ($\tilde{\gamma}_i \in \Gamma_p$, $\gamma \in \Gamma$), so $\xi\gamma \in \Gamma\alpha_i$ for some i, hence $\tilde{\gamma}_1\xi\tilde{\gamma}_2 \in \Gamma_p\alpha_i$. To see $\xi\Gamma_p \subseteq \Gamma_p\xi\Gamma$ is really just a density argument: Let $q = r(\xi)$ be the similitude factor associated to ξ. Recall (see [1]) that there is a natural surjective homomorphism $Sp_n(R) \to Sp_n(R/qR)$ with $R = \mathbb{Z}$ or $\mathbb{Z}_p$. Denote by $\Gamma_p(q)$ the kernel of the homomorphism $Sp_n(\mathbb{Z}_p) \to Sp_n(\mathbb{Z}_p/q\mathbb{Z}_p)$. From Chapter 2, §3.3 of [1], we have that $\Gamma_p(q) \subset \Gamma_p \cap \xi^{-1}\Gamma_p\xi$. Using the fact that $Sp_n(\mathbb{Z}_p/q\mathbb{Z}_p) \cong Sp_n(\mathbb{Z}/q\mathbb{Z})$ (and that $Sp_n(\mathbb{Z}) \to Sp_n(\mathbb{Z}/q\mathbb{Z})$

is surjective), we may write $\Gamma_p = \cup \Gamma_p(q)\delta_i$ with the $\delta_i \in \Gamma$. Let $\tilde{\gamma} \in \Gamma_p$, and write $\tilde{\gamma} = \tilde{\gamma}_0\delta_k$ for some $\tilde{\gamma}_0 \in \Gamma_p(q)$ and some k. Then

$$\xi\tilde{\gamma} = \xi\tilde{\gamma}_0\delta_k = \xi\tilde{\gamma}_0\delta_k(\delta_k^{-1}\xi^{-1})\xi\delta_k = (\xi\tilde{\gamma}_0\xi^{-1})\xi\delta_k \in \Gamma_p\xi\Gamma$$

since $\xi\tilde{\gamma}_0\xi^{-1} \in \xi\Gamma_p(q)\xi^{-1} \subset \xi(\Gamma_p \cap \xi^{-1}\Gamma_p\xi)\xi^{-1} \subset \Gamma_p$.

□

The integral Hecke algebra $\underline{\mathcal{H}}_p$ is generated by the $(n+1)$ Hecke operators

$$T(p) = \Gamma_p \begin{pmatrix} I_n & 0 \\ 0 & pI_n \end{pmatrix} \Gamma_p$$

and for $k = 1, \ldots, n$,

$$T_k^n(p^2) = T_k(p^2) = \Gamma_p \begin{pmatrix} I_{n-k} & 0 & 0 & 0 \\ 0 & pI_k & 0 & 0 \\ 0 & 0 & p^2 I_{n-k} & 0 \\ 0 & 0 & 0 & pI_k \end{pmatrix} \Gamma_p,$$

while the Hecke algebra $\mathcal{H}_p$ is generated by the $(n+1)$ elements above together with the element $T_n(p^2)^{-1} = (pI_{2n})^{-1}$.

The Satake isomorphism shows that the local Hecke algebra is isomorphic to a polynomial ring invariant under a Weyl group:

$$\begin{aligned} &\underline{\mathcal{H}}_p \cong \mathbb{Q}[x_0, \ldots, x_n]^{W_n} \\ &\mathcal{H}_p \cong \mathbb{Q}[x_0^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n} \cong \mathbb{Q}[x_0, \ldots, x_n]^{W_n}[(x_0^2 x_1 \cdots x_n)^{-1}] \end{aligned}$$

where W_n is the group of $\mathbb{Q}$-automorphisms of the rational function field $\mathbb{Q}(x_0, \ldots, x_n)$ generated by all permutations of the variables $x_1, \ldots, x_n$ and by the automorphisms $\tau_1, \ldots, \tau_n$ which are given by:

$$\tau_i(x_0) = x_0 x_i, \qquad \tau_i(x_i) = x_i^{-1}, \qquad \tau_i(x_j) = x_j \quad (0 < j \neq i).$$

W_n is a signed permutation group, in particular, $W_n = \langle \tau_i \rangle \rtimes S_n \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n \cong C_n$ where C_n is Coxeter group associated to the spherical building for $Sp_n(\mathbb{Q}_p)$.

2.2 Symplectic lattices and elementary divisors

Let E be the field $\mathbb{Q}$ or $\mathbb{Q}_p$, $\mathcal{O}$ its ring of integers, and $(V, \langle *, * \rangle)$ a $2n$-dimensional symplectic space over E. Fix a symplectic basis $\mathcal{B} = \{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ of V satisfying $\langle u_i, v_j \rangle = \delta_{ij}$ (Kronecker delta), $\langle u_i, u_j \rangle = \langle v_i, v_j \rangle = 0$. Let $\Gamma = Sp_n(\mathcal{O})$, and let $G = GSp_n(E)$ be the group of symplectic similitudes with scalar factor $r(M) \in E^\times$. Note that Γ is a normal subgroup of $GSp_n(E)$, being the kernel of the determinant map to $E^\times$.

Notation 2.2. *We shall have need of the notion of symplectic divisors (elementary divisors with respect to the symplectic group). To that end, let S denote a set of representatives of $E^\times/\mathcal{O}^\times$. For $E = \mathbb{Q}$, we let $S = \mathbb{Q}_+^\times$, the positive rationals, while for $E = \mathbb{Q}_p$, we let $S = \{p^\nu \mid \nu \in \mathbb{Z}\}$. We will denote by $GSp_n^S(E) = \{M \in GSp_n(E) \mid r(M) \in S\}$, so when $E = \mathbb{Q}$, $GSp_n^S(E)$ is the classical $GSp_n^+(\mathbb{Q})$. It is worth noting that $Sp_n(E) \subset GSp_n^S(E)$.*

With obvious modification to the proof, the following is Lemma 3.6 of [1].

Lemma 2.3. *Let $\xi \in GSp_n^S(E)$, then every double coset $\Gamma\xi\Gamma$ has a unique representative of the form $sd(\xi) = \operatorname{diag}(d_1, \dots, d_n, e_1, \dots, e_n)$ where $d_i, e_i \in S$ and $d_i \mid d_{i+1}$, $d_n \mid e_n$, $e_{i+1} \mid e_i$, and $d_i e_i = r(\xi)$*

We call a lattice symplectic if it has an $\mathcal{O}$-basis which is a symplectic basis for V with respect to the alternating bilinear form on V.

Proposition 2.4. *Let $\mathcal{L}$ be a symplectic lattice. Then $\Gamma = \{A \in GSp_n^S(E) \mid \mathcal{L}A = \mathcal{L}\}$, where the action of A on $\mathcal{L}$ is to be considered as the matrix of a linear transformation with respect to a fixed basis of $\mathcal{L}$.*

Proof. Let $\widetilde{\Gamma} = \{A \in GSp_n^S(E) \mid \mathcal{L}A = \mathcal{L}\}$. It is clear that $\Gamma \subset \widetilde{\Gamma}$, so we need only establish the other inclusion. Let $A \in \widetilde{\Gamma}$. Since $\widetilde{\Gamma}$ is a group, and $\Gamma \subset \widetilde{\Gamma}$, any element of the double coset $\Gamma A \Gamma$ is also an element of $\widetilde{\Gamma}$. By Lemma 2.3, we have that $sd(A) = \operatorname{diag}(d_1, \dots, d_n; e_1, \dots, e_n) \in \widetilde{\Gamma}$, where the $d_i, e_i \in S$ satisfy $e_1\mathcal{O} \subset \cdots \subset e_n\mathcal{O} \subset d_n\mathcal{O} \subset \cdots \subset d_1\mathcal{O}$ and $e_i d_i = r(A)$. It is trivial to see that the elementary divisors of $\mathcal{L}sd(A)$ in $\mathcal{L}$ are $\{\mathcal{L} : \mathcal{L}sd(A)\} = \{d_1, \dots, d_n, e_1, \dots e_n\}$. On the other hand, $\mathcal{L}A = \mathcal{L}$, so $\mathcal{L}sd(A) = \mathcal{L}$ and we must have $d_i = e_i = 1$ for all i. Thus $\Gamma A \Gamma = \Gamma$, so $A \in \Gamma$. ☐

Fix a symplectic lattice $\mathcal{L}$ and put $\mathcal{R} = \mathcal{R}_{\mathcal{L}} = \{\mathcal{L}A \mid A \in GSp_n^S(E)\}$.

Lemma 2.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be lattices in $\mathcal{R}$. Then there exists a symplectic basis $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ of V, and elements $d_i, e_i \in S$ satisfying $e_1\mathcal{O} \subset \cdots \subset e_n\mathcal{O} \subset d_n\mathcal{O} \subset \cdots \subset d_1\mathcal{O}$ and $e_i d_i = r \in S$ such that $\mathcal{M} = \bigoplus_{i=1}^n \mathcal{O}u_i \oplus \bigoplus_{i=1}^n \mathcal{O}v_i$ and $\mathcal{N} = \bigoplus_{i=1}^n \mathcal{O}d_i u_i \oplus \bigoplus_{i=1}^n \mathcal{O}e_i v_i$.*

Remark 2.6.
1. *Note that in the general linear case, GSp_n^S would be replaced by GL_{2n}^+, and $\mathcal{R}$ would be the set of all lattices of full rank in V.*
2. *The ideals $d_i\mathcal{O}$ and $e_j\mathcal{O}$ are called the symplectic divisors of $\mathcal{N}$ in $\mathcal{M}$, and coincide with the standard elementary divisors $\{\mathcal{M} : \mathcal{N}\}$ since $\Gamma \subset SL_{2n}(\mathcal{O})$. That is, if we choose two lattices from $\mathcal{R}$ and consider their elementary divisors in the traditional sense, they are in fact symplectic elementary divisors with the additional properties stated above. In particular, if $\mathcal{M}$ and $\mathcal{N}$ are as in the lemma, we will write $\{\mathcal{M} : \mathcal{N}\} = \{d_1, \dots, d_n, e_1, \dots, e_n\}$ to mean there exist bases of $\mathcal{M}$ and $\mathcal{N}$ as in the lemma.*

Proof. Since $\mathcal{M}$ and $\mathcal{N}$ are in $\mathcal{R}$, there exists an $A \in GSp_n^S(E)$ with $\mathcal{N} = \mathcal{M}A$. Assume that Γ is identified with the stabilizer of $\mathcal{M}$. By Lemma 2.3, $sd(A) = \mathrm{diag}(d_1, \ldots, d_n, e_1, \ldots, e_n) = \gamma_1 A \gamma_2$ for some $\gamma_i \in \Gamma$, where $sd(A)$ is the "symplectic divisor" matrix of A. Finally, it is clear that since $\mathcal{M}\gamma_i = \mathcal{M}$, that $\{\mathcal{M} : \mathcal{N}\} = \{\mathcal{M}\gamma_1 : \mathcal{M}\gamma_1 A\} = \{\mathcal{M}\gamma_1\gamma_2 : \mathcal{M}\gamma_1 A\gamma_2\} = \{\mathcal{M} : \mathcal{M}sd(A)\} = \{d_1, \ldots, d_n, e_1, \ldots, e_n\}$, from which the lemma follows. □

Lemma 2.7. *For A and B in $GSP_n^S(E)$, $\Gamma A = \Gamma B$ if and only if $\mathcal{L}A = \mathcal{L}B$.*

Proof. $\Gamma A = \Gamma B$ if and only if $AB^{-1} \in \Gamma$, which by Proposition 2.4 is true if and only if $\mathcal{L} = \mathcal{L}AB^{-1}$. □

Lemma 2.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be lattices in $\mathcal{R}$. The elementary divisors of $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{L}$ satisfy $\{\mathcal{L} : \mathcal{M}\} = \{\mathcal{L} : \mathcal{N}\}$ if and only if there exists an $A \in \Gamma$ such that $\mathcal{M}A = \mathcal{N}$.*

Proof. The result is clear if there exists an $A \in \Gamma$ such that $\mathcal{M}A = \mathcal{N}$. To prove the converse, we note that by definition of the symplectic elementary divisors, there exist elements d_i, $e_i \in S$ satisfying $e_1\mathcal{O} \subset \cdots \subset e_n\mathcal{O} \subset d_n\mathcal{O} \subset \cdots \subset d_1\mathcal{O}$ and $e_i d_i = r \in S$ and symplectic $\mathcal{O}$-bases

$$\{u_1^{(j)}, \ldots, u_n^{(j)}; v_1^{(j)}, \ldots, v_n^{(j)}\} \qquad (j = 1, 2)$$

of $\mathcal{L}$ such that

$$\begin{aligned} \mathcal{L} &= \bigoplus_{i=1}^n \mathcal{O}u_i^{(1)} \oplus \bigoplus_{i=1}^n \mathcal{O}v_i^{(1)} & \mathcal{M} &= \bigoplus_{i=1}^n \mathcal{O}d_i u_i^{(1)} \oplus \bigoplus_{i=1}^n \mathcal{O}e_i v_i^{(1)} \\ \mathcal{L} &= \bigoplus_{i=1}^n \mathcal{O}u_i^{(2)} \oplus \bigoplus_{i=1}^n \mathcal{O}v_i^{(2)} & \mathcal{N} &= \bigoplus_{i=1}^n \mathcal{O}d_i u_i^{(2)} \oplus \bigoplus_{i=1}^n \mathcal{O}e_i v_i^{(2)} \end{aligned}$$

Let A be the matrix of the linear transformation (with respect to either basis) taking $u_i^{(1)} \mapsto u_i^{(2)}$, and $v_i^{(1)} \mapsto v_i^{(2)}$. Clearly $A \in Sp_n(E) \subset GSp_n^S(E)$ as it maps one symplectic basis to another. Since $\mathcal{L}A = \mathcal{L}$, $A \in \Gamma$ by Proposition 2.4 above. Since A obviously maps $\mathcal{M}$ to $\mathcal{N}$, the proof is complete. □

Proposition 2.9. *Let $A \in GSp_n^S(E)$, and*

$$\Gamma A \Gamma = \Gamma sd(A) \Gamma = \Gamma\, \mathrm{diag}(d_1, \ldots, d_n, e_1, \ldots, e_n)\Gamma.$$

Then $\Gamma\xi \mapsto \mathcal{L}\xi$ gives a one-to-one correspondence between the cosets $\Gamma\xi$ in $\Gamma A\Gamma$ and lattices $M \in \mathcal{R}$ with $\{\mathcal{L} : \mathcal{M}\} = \{d_1, \ldots, d_n, e_1, \ldots e_n\}$.

Proof. We may assume that $A = \mathrm{diag}(d_1, \ldots, d_n, e_1, \ldots, e_n)$. If $\Gamma\xi = \Gamma A\delta$ with $\delta \in \Gamma$, then $\mathcal{L}\xi \in \mathcal{R}$ and we have $\{\mathcal{L} : \mathcal{L}\xi\} = \{\mathcal{L} : \mathcal{L}A\delta\} = \{\mathcal{L} : \mathcal{L}A\} = \{d_1, \ldots, d_n, e_1, \ldots e_n\}$. Conversely, if $\mathcal{M} \in \mathcal{R}$ and $\{\mathcal{L} : \mathcal{M}\} = \{d_1, \ldots, d_n, e_1, \ldots e_n\}$, then by Lemma 2.8 there exists an element $B \in \Gamma$ such that $\mathcal{M} = \mathcal{L}AB$. Clearly $\Gamma AB \subset \Gamma A\Gamma$. The correspondence is one-to-one since by Lemma 2.7, $\Gamma\xi = \Gamma\zeta$ if and only if $\mathcal{L}\xi = \mathcal{L}\zeta$. □

3 The Bruhat-Tits Building for $Sp_n(\mathbb{Q}_p)$

3.1 Lattices

The Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$ is an n-dimensional simplicial complex whose vertices are homothety classes of lattices in a fixed symplectic vector space V. One defines an incidence relation on the vertices, and the resulting flag complex is the building. Generally our interest will be in an apartment in the building for which we need a careful understanding of how the vertices are indexed by classes of lattices, and later how the special vertices are labeled by monomials in $(n+1)$ variables. Some of the basic material can be found in Chapter 20 of [5]; we supplement where germane. Most of this can be done over any field with a discrete valuation, so we temporarily adopt this more general setting.

Let K be a field with a discrete valuation, $\mathcal{O}$ the valuation ring, π a uniformizing parameter, and $k = \mathcal{O}/\pi O$ the residue field. Let $(V, \langle *, * \rangle)$ be a symplectic (non-degenerate alternating) space of dimension $2n$.

Definition 3.1. *An $\mathcal{O}$-lattice Λ is primitive if $\langle \Lambda, \Lambda \rangle \subseteq \mathcal{O}$ and $\langle *, * \rangle$ induces a non-degenerate form on the alternating space $\Lambda/\pi\Lambda$ over k.*

We describe an apartment system for the building as follows. A *frame* is an unordered n-tuple $\{\lambda_1^1, \lambda_1^2\}, \ldots, \{\lambda_n^1, \lambda_n^2\}$ of pairs of lines $\{\lambda_i^1, \lambda_i^2\}$ so that $V = \sum_1^n (\lambda_i^1 + \lambda_i^2)$, $(\lambda_i^1 + \lambda_i^2)$ is orthogonal to $(\lambda_j^1 + \lambda_j^2)$ for $i \neq j$, and each $(\lambda_i^1 + \lambda_i^2)$ is a hyperbolic plane. We say that the frame determines the apartment Σ. Vertices in Σ are homothety classes of lattices, denoted $[\Lambda]$. A vertex $[\Lambda]$ lies in Σ (determined by the above frame), if there are free $\mathcal{O}$-modules $M_i^j \subset \lambda_i^j$ so that $\Lambda = \oplus_{i,j} M_i^j$ for some (and hence every) representative Λ of the homothety class. More concretely, vertices of the building are homothety classes of lattices $[\Lambda]$ which possess a representative Λ such that

There exists a lattice Λ_0 with $\pi^{-1}\Lambda_0$ primitive,

$\Lambda_0 \subseteq \Lambda \subseteq \pi^{-1}\Lambda_0$, and

$\langle \Lambda, \Lambda \rangle \subseteq \pi\mathcal{O}$

or equivalently, Λ/Λ_0 is a totally isotropic k-subspace of the non-degenerate alternating space $\pi^{-1}\Lambda_0/\Lambda_0$.

The maximal simplicies (chambers) are unordered $(n+1)$-tuples $[\Lambda_0], [\Lambda_1] \ldots, [\Lambda_n]$ of homothety classes of lattices with representatives Λ_i satisfying:

$\pi^{-1}\Lambda_0$ is primitive,

$\Lambda_0 \subseteq \Lambda_i \subseteq \pi^{-1}\Lambda_0$, and

$\Lambda_1/\Lambda_0 \subset \Lambda_2/\Lambda_0 \subset \cdots \subset \Lambda_n/\Lambda_0$ is a maximal isotropic flags of k-subspaces in $\pi^{-1}\Lambda_0/\Lambda_0$.

With respect to a fixed symplectic basis $\{e_1, f_1, \ldots, e_n, f_n\}$ ($\langle e_i, f_i \rangle = 1$, $\langle e_i, e_i \rangle = \langle f_i, f_i \rangle = 0$), let Λ be the $\mathcal{O}$-lattice $\Lambda = \mathcal{O}\pi^{a_1}e_1 \oplus \cdots \mathcal{O}\pi^{a_n}e_n \oplus \mathcal{O}\pi^{b_1}f_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}f_n$. With the basis fixed, we often denote this lattice as $(\pi^{a_1}, \ldots \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n})$.

We note that $\langle \Lambda, \Lambda \rangle \subseteq \mathcal{O}$ iff $\langle \pi^{a_i} e_i, \pi^{b_i} f_i \rangle = \pi^{a_i+b_i} \in \mathcal{O}$ which is true iff $a_i + b_i \geq 0$. Given $a_i + b_i \geq 0$, the induced alternating form on $\Lambda/\pi\Lambda$ is non-degenerate over $k = \mathcal{O}/\pi\mathcal{O}$ iff $a_i + b_i = 0$ for all i.

Example 3.2. *Let $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ be a symplectic basis for V (with $\langle e_i, f_i \rangle = 1$), and put $\lambda_i^1 = Ke_i$ and $\lambda_i^2 = Kf_i$. The frame $\{\lambda_i^1, \lambda_i^2\}$ defines an apartment Σ. Let $\Lambda_0 = \pi(\oplus \mathcal{O} e_i \oplus \mathcal{O} f_i)$. Then $\pi^{-1}\Lambda_0$ is primitive. Denote by $[\pi^{a_1}, \dots, \pi^{a_n}; \pi^{b_1}, \dots, \pi^{b_n}]$ the class of the lattice $\mathcal{O}\pi^{a_1} e_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n} e_n \oplus \mathcal{O}\pi^{b_1} f_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n} f_n$. Then the following flags determine (fundamental) chambers in Σ.*

$[\Lambda_0] = [\pi, \dots, \pi; \pi, \dots, \pi] \subset [\Lambda_1] = [1, \pi, \dots, \pi; \pi, \dots, \pi] \subset [\Lambda_2] = [1, 1, \pi, \dots, \pi; \pi, \dots, \pi] \subset \cdots \subset [\Lambda_n] = [1, 1, \dots, 1; \pi, \dots, \pi]$.

$[\Lambda_0] = [\pi, \dots, \pi; \pi, \dots, \pi] \subset [\Lambda_1] = [\pi, \dots, \pi; 1, \pi, \dots, \pi] \subset [\Lambda_2] = [\pi, \dots, \pi; 1, 1, \pi \dots, \pi] \subset \cdots \subset [\Lambda_n] = [\pi, \dots, \pi; 1, 1, \dots, 1]$.

Now to define the building, we start with the set of vertices S which is the set of homothety classes of lattices defined above. We define an incidence relation of S as follows:

Let $s, s' \in S$. We say $s \sim s'$ if there are lattices $\Lambda_s \in s$ and $\Lambda_{s'} \in s'$ and a lattice Λ_0 such that $\pi^{-1}\Lambda_0$ is primitive, $\Lambda_0 \subseteq \Lambda_s \subseteq \pi^{-1}\Lambda_0$, $\pi\Lambda_0 \subseteq \Lambda_{s'} \subseteq \pi^{-1}\Lambda_0$, and either $\Lambda_s \subset \Lambda_{s'}$ or $\Lambda_{s'} \subset \Lambda_s$. The associated flag complex yields the building.

Remark 3.3. *This definition is somewhat subtle. For example, consider the case $n = 2$. Then it would appear if $\Lambda_0 = \pi(\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{O}f_1 \oplus \mathcal{O}f_2)$ that any lattices Λ_s and $\Lambda_{s'}$ satisfying $\pi\Lambda_0 \subset \Lambda_{s'} \subset \Lambda_s \subset \pi^{-1}\Lambda_0$ would give rise to incident vertices s and s'. One must be careful to remember the definition of a vertex, in this case that there must exist an integer ν and a lattice Λ_1 with $\pi^{-1}\Lambda_1$ primitive and $\Lambda_1 \subset \pi^\nu \Lambda_{s'} \subset \pi^{-1}\Lambda_1$ which eliminates some of the possibilities.*

Example 3.4. *Consider the case $n = 2$, with $\Lambda_0 = \pi(\mathcal{O}e_1 \oplus \mathcal{O}e_2 \oplus \mathcal{O}f_1 \oplus \mathcal{O}f_2)$, and let $[\Lambda_0]$ denote any special vertex in the apartment Σ (a vertex with the most hyperplanes through it)*

We restrict our attention to the apartment Σ and hence to the frame determined by the lines spanned by the e_i and f_i. As above, any vertex in Σ is the class of a lattice of the form $\mathcal{O}\pi^{a_1} e_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n} e_n \oplus \mathcal{O}\pi^{b_1} f_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n} f_n$, denoted $[\pi^{a_1}, \dots, \pi^{a_n}; \pi^{b_1}, \dots, \pi^{b_n}]$.

For $n = 2$, we need to provide pairs of lattices Λ_1 and Λ_2 with $\Lambda_1/\Lambda_0 \subset \Lambda_2/\Lambda_0$ a maximal isotropic flag in $\pi^{-1}\Lambda_0/\Lambda_0$, so we only list the pairs of lattices Λ_1 and Λ_2.

Since Sp_n is of type C_n, the Weyl group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$ (the signed permutation group) and has order $2^n n!$, so for $n = 2$ we expect 8 chambers containing the given vertex $= [\Lambda_0] = [1, 1, 1, 1] = [\pi, \pi, \pi, \pi]$. The other pairs of vertices defining the chambers are:

1. $[1, \pi, \pi, \pi] \subset [1, 1, \pi, \pi]$
2. $[\pi, 1, \pi, \pi] \subset [1, 1, \pi, \pi]$
3. $[\pi, \pi, 1, \pi] \subset [\pi, \pi, 1, 1]$

4. $[\pi, \pi, \pi, 1] \subset [\pi, \pi, 1, 1]$

5. $[1, \pi, \pi, \pi] \subset [1, \pi, \pi, 1]$

6. $[\pi, \pi, \pi, 1] \subset [1, \pi, \pi, 1]$

7. $[\pi, 1, \pi, \pi] \subset [\pi, 1, 1, \pi]$

8. $[\pi, \pi, 1, \pi] \subset [\pi, 1, 1, \pi]$

To see how the rest of the apartment is laid out, one must understand the action of the reflections which generate the Weyl group on the lattices. In this context (in contrast to considering the residue of a vertex), we must contend with the affine Weyl group associated to the Bruhat-Tits building for $Sp_n(K)$. The affine Weyl group is of type $\widetilde{C}_n$ which has Coxeter diagram:

$$\bullet_1 \;=\!=\!=\; \bullet_2 \;\text{———}\; \bullet_3 \;\cdots\cdots\; \bullet_{n-1} \;\text{———}\; \bullet_n \;=\!=\!=\; \bullet_{n+1}$$

with $(n+1)$ vertices, and the two endpoints being "special" vertices in the sense of [5]. The Coxeter diagram for C_n is the same with the last special vertex (and associated "edge") deleted. Associated to each vertex i is a reflection s_i, and the reflections satisfy the standard rules $s_i^2 = 1$, $s_i s_j$ has order m_{ij} indicated by the Coxeter diagram ($m_{12} = m_{n(n+1)} = 4$, $m_{i(i+1)} = 3, i \neq 1, n$, and $m_{ij} = 2$ otherwise).

Acting on the symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$, define the reflections (any basis vector not specified is fixed):

- s_1: Interchange e_1 and f_1
- s_j $(2 \leq j \leq n)$: Interchange $e_{j-1} \leftrightarrow e_j$ and $f_{j-1} \leftrightarrow f_j$
- s_{n+1}: $e_n \mapsto \pi^{-1} f_n$, $f_n \mapsto \pi e_n$

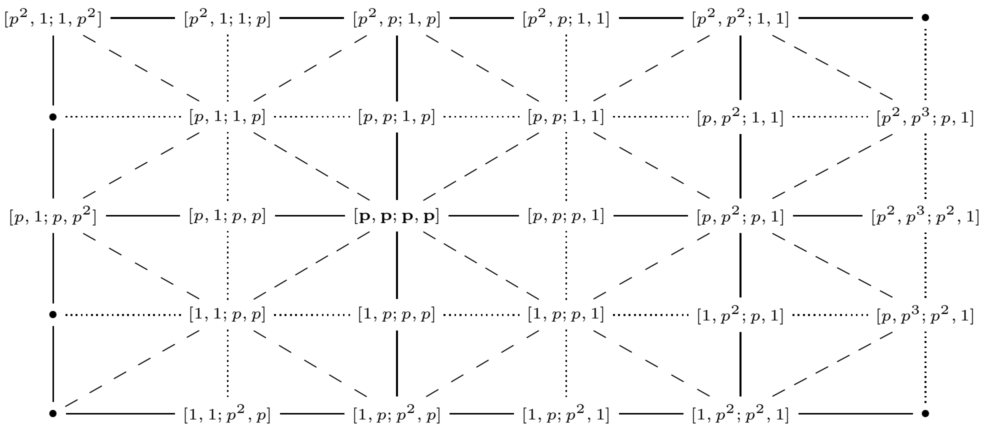
That is, acting on a vertex $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$,

- s_1 takes $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$ to $[\pi^{b_1}, \pi^{a_2}, \ldots, \pi^{a_n}; \pi^{a_1}, \pi^{b_2} \ldots, \pi^{b_n}]$
- s_j $(2 \leq j \leq n)$: takes $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$ to $[\pi^{a_1}, \ldots, \pi^{a_{j-1}}, \pi^{a_j}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_{j-1}}, \pi^{b_j}, \ldots, \pi^{b_n}]$
- s_{n+1}: takes $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$ to $[\pi^{a_1}, \ldots, \pi^{a_{n-1}}, \pi^{b_n+1}; \pi^{b_1}, \ldots, \pi^{b_{n-1}}, \pi^{a_n-1}]$

Now let's proceed to label the apartment Σ. Label a special vertex $v_0 = [\pi, \ldots, \pi; \pi, \ldots, \pi]$, and pick a fundamental chamber containing it. Label each of the codimension 1 faces in the chamber containing the fixed vertex $s_1, \ldots, s_n$. Label the remaining codimension 1 face s_{n+1}.

Each chamber in the building contains two special vertices. In the case of the fundamental chamber, one of them is the vertex v_0 (fixed by the reflections $s_1, \ldots, s_n$), and the other is fixed by $s_2, \ldots, s_{n+1}$. From above, we see that the vertex $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}]$ is fixed by s_j $(2 \le j \le n)$ iff $a_{j-1} = a_j$ and $b_{j-1} = b_j$. The vertex is fixed by s_{n+1} iff $a_n = b_n + 1$, thus $a_i = b_i + 1$ for all i, and so the vertex $[\pi^{a_1}, \ldots, \pi^{a_n}; \pi^{b_1}, \ldots, \pi^{b_n}] = [\pi, \ldots, \pi; 1, \ldots, 1]$ is the other special vertex in the fundamental chamber. Now using the reflections, we may label the apartment.

Example 3.5. *For $Sp_2(\mathbb{Q}_p)$ we have the following (partial) labeling of an apartment by classes of lattices:*



Let $\Lambda = \mathcal{O}\pi^{a_1}e_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}e_n \oplus \mathcal{O}\pi^{b_1}f_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}f_n$. The dual lattice $\Lambda^\sharp$ is defined to be $\{v \in V \mid \langle v, \Lambda \rangle \subseteq \mathcal{O}\}$. It too is a lattice, and it it easily seen from the bilinearity of the alternating form that $\Lambda^\sharp = \mathcal{O}\pi^{-b_1}e_1 \oplus \cdots \oplus \mathcal{O}\pi^{-b_n}e_n \oplus \mathcal{O}\pi^{-a_1}f_1 \oplus \cdots \oplus \mathcal{O}\pi^{-a_n}f_n$. It is also clear that $(\pi^\nu \Lambda)^\sharp = \pi^{-\nu}\Lambda^\sharp$, so $[\Lambda^\sharp]$ depends only on $[\Lambda]$, and in particular $[\Lambda] = [\Lambda^\sharp]$ iff $\pi^\mu \Lambda^\sharp = \Lambda$ for some $\mu \in \mathbb{Z}$.

Proposition 3.6. *Let $\Lambda = \mathcal{O}\pi^{a_1}e_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}e_n \oplus \mathcal{O}\pi^{b_1}f_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}f_n$. Then $[\Lambda] = [\Lambda^\sharp]$ iff there exists an integer μ, so that for all i, $a_i + b_i = \mu$. In this case we call the vertex self-dual.*

Proof. Using our explicit characterization of the dual lattice, $[\Lambda] = [\Lambda^\sharp]$ iff there exists an integer μ so that $\pi^\mu \Lambda^\sharp = \Lambda$, that is iff $\pi^\mu \pi^{-b_i} = \pi^{a_i}$ and $\pi^\mu \pi^{-a_i} = \pi^{b_i}$ which is iff $\mu = a_i + b_i$ for all i. ∎

Proposition 3.7. *If $\Lambda = \mathcal{O}\pi^{a_1}e_1 \oplus \cdots \oplus \mathcal{O}\pi^{a_n}e_n \oplus \mathcal{O}\pi^{b_1}f_1 \oplus \cdots \oplus \mathcal{O}\pi^{b_n}f_n$, and the vertex $[\Lambda]$ is self-dual, then it image under the Weyl group is again a self-dual lattice.*

Proof. We need only check this for the generators of the Weyl group, the s_i, and all of these are obvious from the definitions above. ∎

Remark 3.8. *Actually we can say a little more about the action. The two special vertices in our fundamental chamber ($[\pi, \dots, \pi; \pi, \dots, \pi]$ and $[\pi, \dots, \pi; 1, \dots, 1]$) have the value of μ equal (modulo 2) to 0 and 1 respectively. This value of μ is preserved by the action of the Weyl group which acts transitively on the chambers in the apartment.*

It is now also clear that the Weyl group acts transitively on the special vertices in the apartment (in a "type"-preserving way).

3.2 Symmetric polynomials and Hecke Operators

Our goal is to define $(n+1)$ families of Hecke operators (analogous to the $T_k(p^2)$, $T(p)$) whose generating series produce highly structured rational functions, and hence which are arithmetically interesting. In two of the $n+1$ cases, these rational functions correspond to the spinor and standard zeta functions. In the other cases, they are new. In particular, in no case except for $T(p)$ have any generating series for Hecke operators been expressed as rational functions.

We will make the definitions not in the Hecke algebra (defined by double cosets), but in its representation space, the ring of W_n-invariant polynomials. Doing so will produce a correspondence between the local Hecke algebra slightly different from the Satake correspondence, but who generators are more arithmetically distinguished.

We first give a labeling of the special vertices in an apartment of the Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$ by monomials in $\mathbb{Q}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$ which corresponds in a natural way to symplectic divisors of lattices. We recall that the Satake isomorphism gives us that the local rational Hecke algebra is isomorphic to the ring of polynomials $\mathbb{Q}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$ which are invariant under the Weyl group W_n. Actually, our labelling will be in $\mathbb{Q}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]^{W_n}$ modulo the relation $x_0^2 x_1 \cdots x_n = 1$. Since $\mathcal{H}_p \cong \mathbb{Q}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]^{W_n} \cong \mathbb{Q}[x_0, \dots, x_n]^{W_n}[(x_0^2 x_1 \cdots x_n)^{-1}]$ (see [1]), reducing by the relation $x_0^2 x_1 \cdots x_n = 1$ produces a subring of $\mathbb{Q}[x_0, x_1, \dots, x_n]^{W_n} \cong \underline{\mathcal{H}}_p$, the integral local Hecke algebra. Specifically, we are defining a representation of the local Hecke algebra in the polynomial ring in which double cosets $\Gamma p^\nu I_{2n} \Gamma$ act trivially. This is essentially the case when Hecke operators act on automorphic forms. One should also recall that it is only the invertibility of $\Gamma p I_{2n} \Gamma$ which distinguishes $\mathcal{H}_p$ for $\underline{\mathcal{H}}_p$.

Fix a (fundamental) apartment Σ in the building by means of a frame and symplectic basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ as in the previous section. Let $[\Lambda_0]$ be the class of the lattice $\Lambda_0 = \mathbb{Z}_p e_1 \oplus \cdots \oplus \mathbb{Z}_p e_n \oplus \mathbb{Z}_p f_1 \oplus \cdots \oplus \mathbb{Z}_p f_n$, labeling a fixed special vertex in the apartment Σ. From the previous section, we saw that a typical vertex $[\Lambda]$ in Σ is special iff the vertex is self-dual, that is $\Lambda = \mathbb{Z}_p p^{a_1} e_1 \oplus \cdots \oplus \mathbb{Z}_p p^{a_n} e_n \oplus \mathbb{Z}_p p^{b_1} f_1 \oplus \cdots \oplus \mathbb{Z}_p p^{b_n} f_n$ for which there is an integer μ with $\mu = a_i + b_i$ for all i. With this notation, we now have a one-to-one correspondence between the classes of lattices (labeling special vertices), and monomials in $\mathbb{Q}[x_0^{\pm 1}, \dots, x_n^{\pm 1}]$:

$$[p^{a_1}, \dots, p^{a_n}; p^{b_1}, \dots, p^{b_n}] \longleftrightarrow x_0^{\mu} x_1^{a_1} \cdots x_n^{a_n}$$

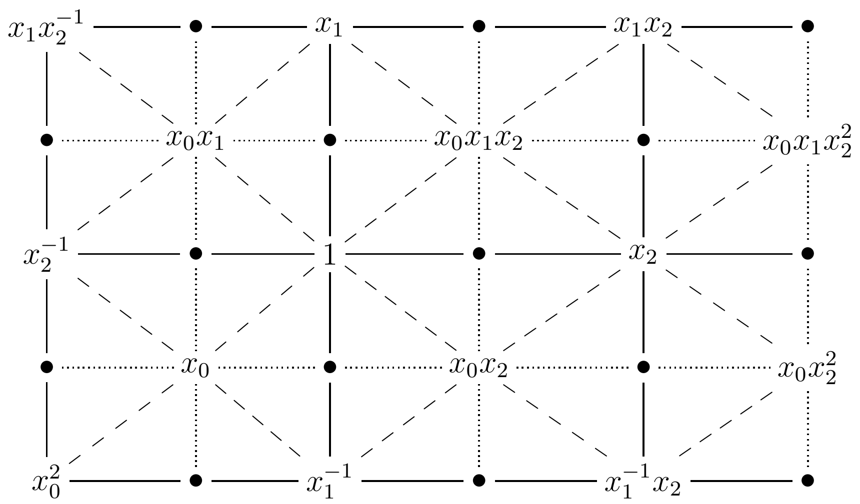
modulo the relation $x_0^2 x_1 \cdots x_n = 1$ which corresponds to the class $[p, \dots, p; p, \dots, p] = [\Lambda_0]$. That is, if Λ is replaced by $p^c \Lambda$, $x_0^{\mu} x_1^{a_1} \cdots x_n^{a_n}$ is replaced by $(x_0^2 x_1 \cdots x_n)^c x_0^{\mu} x_1^{a_1} \cdots x_n^{a_n}$, so that classes of lattices correspond to classes of monomials. To keep the notation from getting too involved, we will simply write $x_0^\mu x_1^{a_1}\cdots x_n^{a_n}$ rather than something like $[x_0^\mu x_1^{a_1}\cdots x_n^{a_n}]$. This avoids obvious confusion in statements like $\mathbb{Q}[x_0^{\pm 1},\ldots,x_n^{\pm 1}]^{W_n} = \mathbb{Q}[x_0,\ldots,x_n]^{W_n}[(x_0^2x_1\cdots x_n)^{-1}]$.

Remark 3.9. *We also note one more important property of this labelling. Any special vertex $v = [p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n}]$ in the apartment can be viewed as an element of $GSp_n^+(\mathbb{Q})$ via $v \leftrightarrow \operatorname{diag}(p^{a_1},\ldots,p^{a_n},p^{b_1},\ldots,p^{b_n})$. As such, there is a natural multiplication of the vertices. If (with v as above), we denote $x_0^\mu x_1^{a_1}\cdots x_n^{a_n}$ by x^v, then we have the simple rule $x^v x^{v'} = x^{vv'}$. We will have need of this observation in the next section.*

We take a moment to foreshadow a bit. While the Satake isomorphism provides a correspondence between the local Hecke algebra and the polynomial ring, the correspondence does not appear at all obvious. On the other hand, with the given notation, there is an obvious correspondence (though not completely correct as yet) with the local Hecke algebra: Given, $[p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n}] \longleftrightarrow x_0^\mu x_1^{a_1}\cdots x_n^{a_n}$ with $\mu = a_i+b_i$, we immediately note that $\operatorname{diag}(p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n}) \in GSp_n^+(\mathbb{Q})$, so that $\Gamma_n \operatorname{diag}(p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n})\Gamma_n$ is in the local Hecke algebra $\mathcal{H}_p$. Thus there is a clear connection between the Hecke operator $\Gamma_n \operatorname{diag}(p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n})\Gamma_n$ and the monomial $x_0^\mu x_1^{a_1}\cdots x_n^{a_n}$ which we will turn into a valid representation.

Before proceeding to develop the representation, we provide a labeling of a piece of the apartment Σ for Sp_2, corresponding to our previous labelling by classes of lattices:

Example 3.10. *A partial labeling of the special vertices in an apartment for $Sp_2(\mathbb{Q}_p)$ by monomials*



Remark 3.11. *We note that if we think of each monomial as acting on the vertex 1 as a motion (e.g. x_2 represents motion two vertices "to the right"), then multiplication by monomials is consistent with this action (e.g., x_1x_2 is two vertices to the right of x_1).*

The motivation behind the definition of the new Hecke operators comes from several sources. On the one hand, Hecke operators, thought of as double cosets, have an action on automorphic forms defined in terms of a "sum" of their right cosets, which by Proposition 2.9 corresponds to a sum of (classes of) lattices with prescribed symplectic divisors.

On the other hand, Hecke operators (via the Satake isomorphism) correspond to W_n-invariant polynomials which correspond to sums of vertices in the building. In fact, we are really thinking only of the underlying graph and not worrying about the full simplicial structure the building embodies. That said, natural operators on graphs are adjacency operators (a sum of certain neighboring vertices), so combining these ideas, we are led in the following direction: A Hecke operator should be a sum of monomials (corresponding to lattices) which are invariant under W_n.

To define our Hecke operators in the context of this polynomial ring we need a definition and simple lemma: For a nonnegative integer ℓ, define $h^r(\ell) = \sum_{\substack{\sum j_k=\ell \\ j_k\geq 0}} z_1^{j_1} z_2^{j_2}\cdots z_r^{j_r}$. Note that $h^r(\ell)$ is a symmetric polynomial in the r variables $z_1,\ldots,z_r$, and in particular, $h^r(0)=1$ and $h^r(1) = z_1+\cdots+z_r$.

Proposition 3.12. *The generating series associated to the $h^r(\ell)$ satisfies*

$$\sum_{\ell\geq 0} h^r(\ell)u^\ell = [(1-uz_1)\cdots(1-uz_r)]^{-1}$$

Proof. This is essentially obvious:

$$\begin{aligned}
[(1-uz_1)\cdots(1-uz_r)]^{-1} &= \left(\sum_{a_1\geq 0}(uz_1)^{a_1}\right)\cdots\left(\sum_{a_r\geq 0}(uz_r)^{a_r}\right) \\
&= \sum_{\ell\geq 0} u^\ell \cdot \Big[\sum_{\substack{\sum a_i=\ell \\ a_i\geq 0}} z_1^{a_1}\cdots z_r^{a_r}\Big]
\end{aligned}$$

It is clear from the definitions above that the coefficient of u^ℓ in the given expression is $h^r(\ell)$. □

Next we need to use the above polynomial to create a W_n-invariant polynomial. The simplest examples are simply to fix a monomial and to sum its images under the action of W_n. To that end, we compute a few simple orbits.

Lemma 3.13. *Under the action of W_n, we obtain the following orbits:*

1. $\mathrm{Orbit}_{W_n}(x_0) = \{x_0 x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n} \mid \varepsilon_i = 0,1\}$.
2. $\mathrm{Orbit}_{W_n}(x_1\cdots x_k) = \{x_{i_1}^{\delta_{i_1}}\cdots x_{i_k}^{\delta_{i_k}} \mid 1\leq i_1\leq\cdots\leq i_k\leq n, \delta_{i_j}=\pm 1\}$.

In particular, the orbits have size 2^n *and* $2^k\binom{n}{k}$ *respectively.*

Proof. Recall that W_n is the group of $\mathbb{Q}$-automorphisms of the rational function field $\mathbb{Q}(x_0,\dots,x_n)$ generated by all permutations of the variables $x_1,\dots,x_n$ and by the automorphisms $\tau_1,\dots,\tau_n$ which are given by:

$$\tau_i(x_0)=x_0x_i,\qquad \tau_i(x_i)=x_i^{-1},\qquad \tau_i(x_j)=x_j\quad (0<j\neq i).$$

In particular, $W_n=\langle\tau_i\rangle\rtimes S_n\cong(\mathbb{Z}/2\mathbb{Z})^n\rtimes S_n$. Since $\tau_{k_s}\tau_{k_{s-1}}\cdots\tau_{k_1}(x_0)=x_0x_{k_1}\cdots x_{k_s}$ for distinct $k_j\geq 1$, it is clear that $\mathrm{Orbit}_{W_n}(x_0)\supseteq\{x_0x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}\mid\varepsilon_i=0,1\}$, and so the orbit has cardinality at least 2^n. On the other hand, all of S_n is contained in the stabilizer of x_0, so the size of the orbit is $[W_n:\mathrm{Stab}(x_0)]\leq[W_n:S_n]=2^n$, which gives the first result.

For the second, it is easy to see directly: S_n can take $x_1\cdots x_k$ to any monomial $x_{i_1}\cdots x_{i_k}$ with $1\leq i_1\leq\cdots\leq i_k\leq n$. Applying τ_{i_j} takes x_{i_j} to $x_{i_j}^{-1}$ fixing all other indices. Since these generate the group W_n, the orbit and its size are clear. □

With these orbits determined, the following definitions become less mysterious. We start with $h^r(\ell)$ where r is the size of one of the above orbits and substitute for the variables z_i the elements in the orbit. Thus we define the families of Hecke operators:

$$t_0^n(p^\ell)=h^{2^n}(\ell)\Big|_{\substack{z_i\mapsto\sigma_i(x_0)\\ \sigma_i\in W_n/\operatorname{Stab}(x_0)}}.$$

and for $1\leq k\leq n$,

$$t_k^n(p^\ell)=h^{2^k\binom{n}{k}}(\ell)\Big|_{\substack{z_i\mapsto\sigma_i(x_1\cdots x_k)\\ \sigma_i\in W_n/\operatorname{Stab}(x_1\cdots x_k)}}.$$

In particular,

$$t_0^n(p)=\sum_{\varepsilon_i=0,1}x_0x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}.\quad (2^n\text{ summands})$$

and

$$t_k^n(p)=\sum_{\substack{1\leq i_1<\cdots<i_k\leq n\\ \delta_{i_j}=\pm1}}x_{i_1}^{\delta_{i_1}}\cdots x_{i_k}^{\delta_{i_k}}.\quad \left(2^k\binom{n}{k}\text{ summands}\right)$$

Here once again, we are suppressing the fact that we are looking at classes of monomials, and that really we have defined (for example)

$$t_0^n(p)=\sum_{\varepsilon_i=0,1}[x_0x_1^{\varepsilon_1}\cdots x_n^{\varepsilon_n}].$$

Since $x_0^2x_1\cdots x_n$ is fixed by every element of W_n, these operators are well-defined.

Remark 3.14. *We want to make clear the naturalness of these operators as well as their significance. As alluded to above, $t_j(p)$ represents an adjacency operator. To see this observe, consider the case $n = 2$ and the apartments labeled in diagrams in Examples 3.5 and 3.10.*

$$t_0(p) = x_0 + x_0x_1 + x_0x_2 + x_0x_1x_2 = [1,1;p,p] + [p,1;1,p] + [1,p;p,1] + [p,p;1,1]$$
$$t_1(p) = x_1 + x_1^{-1} + x_2 + x_2^{-1} = [p^2,p;1,p] + [1,p;p^2,p] + [p,p^2;p,1] + [p,1,;p,p^2]$$
$$t_2(p) = x_1x_2 + x_1x_2^{-1} + x_1^{-1}x_2 + x_1^{-1}x_2^{-1} = [p^2,p^2;1,1] + [p^2,1;1,p^2] + [1,p^2;p^2,1] + [1,1;p^2,p^2]$$

Thus at least restricted to the apartment we have the following natural correspondences:

$$t_0(p) = \sum_{\Gamma M\Gamma = \Gamma\,\mathrm{diag}(1,1;p,p)\Gamma} [M] \longleftrightarrow T(p)$$
$$t_1(p) = \sum_{\Gamma M\Gamma = \Gamma\,\mathrm{diag}(1,p;p^2,p)\Gamma} [M] \longleftrightarrow T_1(p^2)$$

and in general $t_k(p) \longleftrightarrow T^n_{n-k}(p^2)$ for $1 \le k < n$ ($T_n(p^2) \longleftrightarrow x_0^2x_1\cdots x_n$).

To see the significance of these operators we need to provide a little background. Recall that associated to a simultaneous Hecke eigenfunction F of weight k for $Sp_n(\mathbb{Z})$ are the Satake p-parameters $(\alpha_0,\ldots,\alpha_n) = (\alpha_0(p),\ldots,\alpha_n(p)) \in \mathbb{C}^{n+1}/W_n$ for each prime p, generalizing the Hecke eigenvalues. The Satake parameters satisfy $\alpha_0(p)^2\alpha_1(p)\cdots\alpha_n(p) = p^{nk-n(n+1)/2}$ and are used to define the spinor and standard zeta functions.

The standard zeta function is defined by $D_F(s) = \prod_p D_{F,p}(p^{-s})^{-1}$ ($\Re(s) > 1$), where

$$D_{F,p}(v) = (1-v)\prod_{m=1}^{n}(1-\alpha_m v)(1-\alpha_m^{-1}v),$$

while the spinor zeta function is defined by $Z_F(s) = \prod_p Z_{F,p}(p^{-s})^{-1}$ ($\Re(s) > nk/2 - n(n+1)/4 + 1$), where

$$Z_{F,p}(v) = (1-\alpha_0 v)\prod_{m=1}^{n}\prod_{1\le i_1<\cdots<i_m\le n}(1-\alpha_0\alpha_{i_1}\cdots\alpha_{i_m}v).$$

For Sp_2, Andrianov and Zhuravlev [1] define a family of Hecke operators $T^2(p^\ell)$ whose images under the (Satake) spherical map Ω (from $\mathcal{H}_p$ to $\mathbb{Q}[x_0^{\pm1},x_1^{\pm1},x_2^{\pm1}]^{W_2}$) satisfy

$$\sum_{\ell\ge0}\Omega(T^2(p^\ell))v^\ell = \frac{(1-p^{-1}x_0^2x_1x_2v^2)}{(1-x_0v)(1-x_0x_1v)(1-x_0x_2v)(1-x_0x_1x_2v)},$$

the denominator of which is essentially that of the spinor zeta function for Sp_2:

$Z_F(s) = \prod Z_{F,p}(p^{-s})^{-1}$, where $Z_{F,p}(v) = (1-\alpha_0 v)(1-\alpha_0\alpha_1 v)(1-\alpha_0\alpha_2 v)(1-\alpha_0\alpha_1\alpha_2 v)$. We have taken the opposite perspective here. We have defined interesting operators in the polynomial ring and will determine what they correspond to in the Hecke algebra.

Proposition 3.15. *The operators $t_0(p^\ell)$ and $t_k(p^\ell)$ have a generating series which is a rational function:*

$$\sum_{\ell\geq 0} t_0^n(p^\ell)v^\ell = \left[(1-x_0 v)\prod_{m=1}^{n}\prod_{1\leq i_1<\cdots<i_m\leq n}(1-x_0 x_{i_1}\cdots x_{i_m} v)\right]^{-1}$$

which clearly corresponds to the local factor of the spinor zeta function, and

$$\sum_{\ell\geq 0} t_k^n(p^\ell)v^\ell = \left[\prod_{1\leq i_1<\cdots<i_k\leq n,\ \delta_{i_j}=\pm 1}(1-x_{i_1}^{\delta_{i_1}}\cdots x_{i_k}^{\delta_{i_k}} v)\right]^{-1}$$

which, when $k=1$, is simply $\sum_{\ell\geq 0} t_1^n(p^\ell)v^\ell = \left[\prod_{m=1}^{n}(1-x_m v)(1-x_m^{-1}v)\right]^{-1}$ which in turn (up to an initial "zeta" factor) corresponds to the local factor of the standard zeta function: $D_{F,p}(v) = (1-v)\prod_{m=1}^{n}(1-\alpha_m v)(1-\alpha_m^{-1}v)$.

Proof. The proof is immediate from Proposition 3.12 and the computation of orbits in Lemma 3.13. □

Remark 3.16. *Except for $k=0$ and $k=1$, these are new Hecke operators whose generating series are rational functions which likely correspond to new zeta functions which can be studied in the context of Siegel modular forms.*

We note that for the case of $n=2$, the operator $t_0(p^\ell)$ has a generating function with the same denominator as $\Omega(T^2(p^\ell))$, but with numerator 1.

4 Connections with the global Hecke algebra

Recall that we have fixed a $2n$-dimensional symplectic space $(V, \langle *, *\rangle)$ over $\mathbb{Q}$. Let Δ_n the Bruhat-Tits building for $Sp_n(\mathbb{Q}_p)$. Then the vertices of the building, $\mathrm{Vert}(\Delta_n)$, are in one-to-one correspondence with homothety classes of lattices in V. Let $\mathcal{B}$ be the rational vector space with basis $\mathrm{Vert}(\Delta_n)$. In this section, we define two natural (essentially) faithful representations of the local Hecke algebra $\mathcal{H}_p$, and a connecting homomorphism which adds the necessary structure to our intuitive definitions given earlier. We will produce the following commutative diagram of algebra homomorphisms.

$$\begin{tikzcd} \mathcal{H}_p \arrow[r, "\Phi"] \arrow[dr, "\Psi"'] & \mathbb{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n} \arrow[d, "\rho"] \\ & \mathrm{End}(\mathcal{B}) \end{tikzcd}$$

4.1 An action of the Hecke algebra on the building

For $\xi \in GSp_n^+(\mathbb{Q})$, any double coset $\Gamma\xi\Gamma$ can be represented by a diagonal element, so we assume that henceforth. For $\xi = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n}; p^{b_1}, \ldots, p^{b_n}) \in GSp_n^+(\mathbb{Q})$, we have that the double coset $\Gamma\xi\Gamma$ determines a collection of right cosets $\{\Gamma\xi_\nu\}$ which by Proposition 2.9 are in one-to-one correspondence with the collection of lattices $\{M\}$ with $\{L : M\} = \{p^{a_1}, \ldots, p^{a_n}; p^{b_1}, \ldots, p^{b_n}\}$. Using this identification, it is natural to define the operator $T_{\mathcal{B}}(p^{a_1}, \ldots, p^{a_n}; p^{b_1}, \ldots, p^{b_n}) \in \mathrm{End}(\mathcal{B})$ induced by:

$$T_{\mathcal{B}}(p^{a_1}, \ldots, p^{a_n}; p^{b_1}, \ldots, p^{b_n})([L]) = \sum_{\{L:M\}=\{p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n}\}} [M]$$

where the sum is over all vertices in the building with prescibed "elementary divisors".

For brevity, we shall often simply write $T_{\mathcal{B}}(\xi)([L]) = \sum_{\{L:M\}=\xi}[M]$. The map is clearly well-defined and (by definition) linear.

In the usual notation, let $\xi_1 = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n}, p^{b_1}, \ldots, p^{b_n})$, $\xi_2 = \mathrm{diag}(p^{c_1}, \ldots, p^{c_n}, p^{d_1}, \ldots, p^{d_n})$ be elements of $GSp_n^+(\mathbb{Q})$ and write $\Gamma\xi_1\Gamma$ as the disjoint union $\cup\Gamma\alpha_i$, and write $\Gamma\xi_2\Gamma$ as the disjoint union $\cup\Gamma\beta_j$. In the Hecke algebra $\mathcal{H}_p$, the multiplication law is defined by (e.g., see section 3.1 of [6]):

$$(\Gamma\xi_1\Gamma)(\Gamma\xi_2\Gamma) = \Gamma\xi_1\Gamma\xi_2\Gamma = \sum_{i,j} \Gamma\alpha_i\beta_j$$

where the right cosets are not necessarily distinct. More precisely,

$$(\Gamma\xi_1\Gamma)(\Gamma\xi_2\Gamma) = \sum_{i,j} \Gamma\alpha_i\beta_j = \sum_{\Gamma\xi\Gamma} c(\xi)\Gamma\xi\Gamma$$

where the sum is over all double cosets $\Gamma\xi\Gamma \subset \Gamma\xi_1\Gamma\xi_2\Gamma$, and where $c(\xi)$ is the number of pairs (i, j) for which $\Gamma\alpha_i\beta_j = \Gamma\xi$.

Theorem 4.1. *The correspondence $\Gamma\xi\Gamma \mapsto T_{\mathcal{B}}(\xi)$ induces a representation $\Psi : \mathcal{H}_p \to \mathrm{End}(\mathcal{B})$, whose kernel consists of double cosets of the form $\Gamma\xi\Gamma$ with $\xi = p^\mu I_{2n}$, $\mu \in \mathbb{Z}$.*

Proof. We first verify that Ψ is a ring homomorphism. Using the notation above, we have

$$\begin{aligned}T_{\mathcal{B}}(\xi_1)T_{\mathcal{B}}(\xi_2)([L]) &= T_{\mathcal{B}}(\xi_1)(\sum_{\{L:M\}=\xi_2}[M]) \\ &= \sum_{\{L:M\}=\xi_2}\sum_{\{M:N\}=\xi_1}[N]\end{aligned}$$

By Proposition 2.9, each lattice M for which $\{L : M\} = \xi_2$ is of the form $M = L\beta_j$. Now

$$\{M : N\} = \xi_1 \iff \{L\beta_j : N\} = \xi_1 \iff \{L : N\beta_j^{-1}\} = \xi_1$$

Now let P be such that $\{L : P\} = \xi_1$. Then again by Proposition 2.9, $P = L\alpha_i$ for some i. But then $P = N\beta_j^{-1}$, so $N = P\beta_j = L\alpha_i\beta_j$.

Thus, $T_{\mathcal{B}}(\xi_1)T_{\mathcal{B}}(\xi_2)([L]) = \sum_{\{L:M\}=\xi_2}\sum_{\{M:N\}=\xi_1}[N] = \sum_{i,j}[L\alpha_i\beta_j]$. From the discussion preceding the theorem (and once again Proposition 2.9), this last sum is exactly $\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)([L])$ which is the image of $(\Gamma\xi_1\Gamma)(\Gamma\xi_2\Gamma)$.

To compute the kernel of Ψ, suppose $\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)$ is the trivial map. Then

$$\sum_{\Gamma\xi\Gamma} c(\xi)T_{\mathcal{B}}(\xi)([L]) = \sum_{\Gamma\xi\Gamma} c(\xi)\sum_{\{L:M\}=\xi}[M] = [L]$$

for all vertices $[L] \in \mathrm{Vert}(\Delta_n)$. But the elements $[M] \in \mathrm{Vert}(\Delta_n)$ are a basis for $\mathcal{B}$, we have all the only one ξ, and for that ξ, $c(\xi) = 1$. Thus we have $\sum_{\{L:M\}=\xi}[M] = [L]$ for all $[L]$. Now if $\Gamma\xi\Gamma = \cup\xi_\nu$, then by proposition 2.9, $\sum_{\{L:M\}=\xi}[M] = \sum_\nu[L\xi_\nu] = [L]$, so there can be only one right coset: $\Gamma\xi\Gamma = \Gamma\xi$, and $[L\xi] = [L]$. Since $\{L : L\xi\} = \xi$, we must have $\xi = p^\mu I_{2n}$ for some integer μ. $\square$

Now to define the representation $\Phi : \mathcal{H}_p \to \mathbb{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n}$, we begin by defining certain symmetric polynomials. As before, let $\xi = \mathrm{diag}(p^{a_1}, \ldots, p^{a_n}, p^{b_1}, \ldots, p^{b_n}) \in GSp_n^+(\mathbb{Q})$. Recall this means there is an integer μ so that $\mu = a_i + b_i$ for all i. Define $t(\xi)$ to be the sum of the images of $(x_0^\mu x_1^{a_1}\cdots x_n^{a_n})$ under the action of the (spherical Weyl) group W_n. We denote this as

$$t(\xi) = \sum_{w\in W_n}(x_0^\mu x_1^{a_1}\cdots x_n^{a_n})^w$$

Once again, we remind the reader that this sum is really over the classes of monomials $x_0^\mu x_1^{a_1}\cdots x_n^{a_n}$ (modulo $x_0^2 x_1\cdots x_n = 1$) under the action of the Weyl group W_n. Since $x_0^2x_1\cdots x_n$ is fixed by all elements of W_n, this definition is well-defined.

Loosely, speaking Φ should map $\Gamma\xi\Gamma$ to a multiple of $t(\xi)$. The key to defining this homomorphism is the following identification. We know that $W_n \cong C_n$ is simply signed a permutation group (permuting the n pairs of lines defining a frame), and so can be identified with a subgroup of Γ (see 10.1 of [3]).

Remark 4.1. *The isomorphism $W_n \cong C_n$ can be seen quite explicitly if we identify the generators of the affine Weyl group C_n with generators of W_n. In the previous section, we wrote down reflections s_j which generated the Weyl groups: $s_1, \dots, s_n$ generate the spherical Weyl group C_n, while $s_1, \dots, s_{n+1}$ generate the affine Weyl group $\widetilde{C}_n$.*

The group W_n is generated by the τ_i and the permutations in the symmetric group S_n. With obvious identifications, we see that $\tau_j = (i\ j)\tau_i(i\ j)$ and since S_n is generated by the transpositions $\{(1\ 2), (2\ 3), \dots, (n-1\ n)\}$, $W_n = \langle \tau_1, (1\ 2), (2\ 3), \dots, (n-1\ n)\rangle$. One identifies τ_1 with s_1, and $(j-1\ j)$ with s_j $(2 \le j \le n)$ and verifies they satisfy the same Coxeter relations as the s_i, and so satisfy the deletion condition, which by Tit's theorem (see Chapter 1 of [3]) gives a Coxeter system.

By definition,

$$T_{\mathcal{B}}(\xi)([L]) = \sum_{\gamma_\nu \in (\xi^{-1}\Gamma\xi \cap \Gamma)\backslash\Gamma} [L\xi\gamma_\nu]$$

So identifying W_n with a subgroup of Γ, we may write Γ as a union of double cosets: $\Gamma = \cup(\xi^{-1}\Gamma\xi \cap \Gamma)\delta_\mu W_n$, so the expression above becomes

$$T_{\mathcal{B}}(\xi)([L]) = \sum_{\delta_\mu \in (\xi^{-1}\Gamma\xi\cap\Gamma)\backslash\Gamma/W_n} \sum_{w \in W_n} [L\xi\delta_\mu w]$$

Given that W_n acts invariantly on any given apartment, we see that for a fixed μ, the collection of vertices $[L\xi\delta_\mu w]$ as w runs over W_n consists (with multiplicity $[W_n : \mathrm{Stab}(L\xi\delta_\mu)]$) of all the vertices $[M]$ in a given apartment with $\{L : M\} = \xi$. Moreover, since every apartment in the building is isomorphic (with structure determined by the Coxeter group $\widetilde{C}_n$), all of the multiplicities $[W_n : \mathrm{Stab}(L\xi\delta_\mu)]$ are the same, independent of μ.

From the remark above it is now clear that after an appropriate change of frame (in particular a change of a basis), $t(\xi) = \sum_{w\in W_n}(x_0^\mu x_1^{a_1}\cdots x_n^{a_n})^w$ can be naturally identified with summand $\sum_{w\in W_n}[L\xi\delta_\mu w]$ occuring in

$$T_{\mathcal{B}}(\xi)([L]) = \sum_{\delta_\mu \in (\xi^{-1}\Gamma\xi\cap\Gamma)\backslash\Gamma/W_n} \sum_{w \in W_n} [L\xi\delta_\mu w]$$

If we define $\eta(\xi) = \#(\xi^{-1}\Gamma\xi \cap \Gamma)\backslash\Gamma/W_n$, we can identify $T_{\mathcal{B}}(\xi)$ with $\eta(\xi)t(\xi)$, thus we define $\Phi(\Gamma\xi\Gamma) = \eta(\xi)t(\xi)$.

To define the connecting homomorphism $\rho : \mathbb{Q}[x_0^{\pm1}, x_1^{\pm1}, \dots, x_n^{\pm1}]^{W_n} \to \mathrm{End}(\mathcal{B})$, we observe that any polynomial $p \in \mathbb{Q}[x_0^{\pm1}, x_1^{\pm1}, \dots, x_n^{\pm1}]^{W_n}$, can be written as a sum $\sum_\xi c(\xi)t(\xi)$ where the ξ are a set of representatives of the monomials occuring in p modulo W_n, so we only have to define ρ on a basis. We define $\rho(t(\xi)) = \eta(\xi)^{-1}T_{\mathcal{B}}(\xi)$, so that $\rho \circ \Phi = \Psi$.

To prove that Φ is a ring homomorphism, it is convenient to think of Φ as the restriction to $\mathcal{H}_p$ of a linear map $\Phi : L(\Gamma, G) \to \mathbb{Q}[x_0^{\pm1}, x_1^{\pm1}, \dots, x_n^{\pm1}]^{W_n}$. Recall that in this setting we view the double coset $\Gamma\xi\Gamma$ as the sum of right cosets $\sum \Gamma\xi_\nu$ ($\Gamma\xi\Gamma = \cup\Gamma\xi_\nu$) which is right invariant under Γ. While the map Φ (with domain $L(\Gamma, G)$) will only be a vector space homomorphism, its restriction to $\mathcal{H}_p$ will be a ring homomorphism.

What we have said above is that we want Φ to take $\Gamma\xi\Gamma$ to $\eta(\xi)t(\xi)$ so that there are as many monomials (counting multiplicities) as right cosets $\Gamma\xi\Gamma = \cup\Gamma\xi_\nu$. Since any right coset $\Gamma\xi_\nu$ determines the same double coset, we must map any right coset to the same polynomial. From this, the definition is clear: $\Phi(\Gamma\xi_\nu) = |W_n|^{-1}\sum_{w\in W_n}(x^{\xi w})$. Then extending linearly yields the desired $\Phi(\Gamma\xi\Gamma) = \eta(\xi)t(\xi)$.

Theorem 4.2. *The correspondence $\Gamma\xi\Gamma \mapsto \eta(\xi)t(\xi)$ is a surjective homomorphism $\Phi : \mathcal{H}_p \to \mathbb{Q}[x_0^{\pm1}, x_1^{\pm1}, \ldots, x_n^{\pm1}]^{W_n}$, whose kernel consists of double cosets of the form $\Gamma\xi\Gamma$ with $\xi = p^\mu I_{2n}$, $\mu \in \mathbb{Z}$.*

Proof. First note that since $\mathbb{Q}[x_0^{\pm1}, x_1^{\pm1}, \ldots, x_n^{\pm1}]^{W_n} = \mathbb{Q}[x_0, \ldots, x_n]^{W_n}[(x_0^2x_1\cdots x_n)^{-1}]$ (and we are working modulo $x_0^2x_1\cdots x_n = 1$), the codomain is spanned as a vector space by elements of the form $t(\xi)$, from which the surjectivity is immediate. Let $\Gamma\alpha\Gamma, \Gamma\beta\Gamma \in \mathcal{H}_p$, and write (as a disjoint union)

$$\Gamma\alpha\Gamma = \bigcup_{\alpha_i\in(\alpha^{-1}\Gamma\alpha\cap\Gamma)\backslash\Gamma/W_n}\ \bigcup_{w\in W_n}\Gamma\alpha\alpha_i w \text{ and}$$
$$\Gamma\beta\Gamma = \bigcup_{\beta_j\in(\beta^{-1}\Gamma\beta\cap\Gamma)\backslash\Gamma/W_n}\ \bigcup_{w'\in W_n}\Gamma\beta\beta_j w'.$$

Without loss of generality, we are assuming that α and β are diagonal. Then in $\mathcal{H}_p$, $(\Gamma\alpha\Gamma)(\Gamma\beta\Gamma) = \sum_{i,j}\Gamma\alpha\alpha_i w\beta\beta_j w' = \sum_\xi c(\xi)\Gamma\xi\Gamma$. Our first task is to determine the ξ which can occur. Since $\beta_j, w' \in \Gamma$, it is clear that $\Gamma\alpha\alpha_i w\beta\beta_j w'\Gamma = \Gamma\alpha\alpha_i w\beta\Gamma$, however it it also true that $\Gamma\alpha\alpha_i w\beta\Gamma = \Gamma\alpha w\beta\Gamma$. To see the latter we deal with symplectic elementary divisors (Lemma 2.3). Recall that elementary divisors $\{L : L\xi\}$ are uniquely determined by the double coset $\Gamma\xi\Gamma$. We have that

$$\{L : L\alpha\alpha_i w\beta\} = \{L\beta^{-1}w^{-1} : L\alpha\alpha_i\},$$

and we note that $L = L\alpha_i = L\alpha_i^{-1}$ since $\alpha_i \in \Gamma$. Moreover, $\alpha_i^{-1}\alpha\alpha_i$ and α are simply two matrix representations of the same linear transformation acting on L. Thus

$$\{L : L\alpha\alpha_i w\beta\} = \{L\beta^{-1}w^{-1} : L\alpha_i^{-1}\alpha\alpha_i\} = \{L\beta^{-1}w^{-1} : L\alpha\} = \{L : L\alpha w\beta\}$$

Thus $(\Gamma\alpha\Gamma)(\Gamma\beta\Gamma) = \sum_{w\in W_n} d(w)\Gamma\alpha w\beta\Gamma$ (where the double cosets may not be distinct). Here $d(w)$ is easily computed from the above observations:

$$d(w)\deg(\Gamma\alpha w\beta) = \eta(\alpha)\eta(\beta)|W_n|,$$

where $\deg(\Gamma\alpha w\beta)$ is the number of right cosets in the double coset, that is, $\deg(\Gamma\xi\Gamma) = \#\,(\xi^{-1}\Gamma\xi\cap\Gamma)\,\backslash\Gamma$.

It follows that

$$\begin{aligned}\Phi\left((\Gamma\alpha\Gamma)(\Gamma\beta\Gamma)\right) &= \sum_w d(w)\eta(\alpha w\beta)t(\alpha w\beta) \\ &= \sum_w \eta(\alpha)\eta(\beta)|W_n|(\deg(\Gamma\alpha w\beta\Gamma))^{-1}\eta(\alpha w\beta)t(\alpha w\beta) \\ &= \eta(\alpha)\eta(\beta)\sum_w t(\alpha w\beta).\end{aligned}$$

We note that since α and β are diagonal and w is a "signed" permutation, the product $\alpha w\beta$ is diagonal so that $t(\alpha w\beta)$ does indeed make sense. On the other hand

$$\begin{aligned}\Phi(\Gamma\alpha\Gamma)\Phi(\Gamma\beta\Gamma) &= \eta(\alpha)\eta(\beta)t(\alpha)t(\beta) \\ &= \eta(\alpha)\eta(\beta)(\sum_w x^{\alpha w})(\sum_{w'} x^{\beta w'}) \\ &= \eta(\alpha)\eta(\beta)(\sum_w\sum_{w'} x^{\alpha w\beta w'}) \text{ (see remark 3.9)} \\ &= \eta(\alpha)\eta(\beta)\sum_w(\sum_{w'} x^{(\alpha w\beta)w'}) \\ &= \eta(\alpha)\eta(\beta)\sum_w t(\alpha w\beta)\end{aligned}$$

Thus Φ is a ring homomorphism. To compute its kernel, suppose that $\Phi(\sum_\xi c(\xi)\Gamma\xi\Gamma) = \sum_\xi c(\xi)\eta(\xi)t(\xi)$ is trivial, that is to say (as polynomials)

$$\sum_\xi c(\xi)\eta(\xi)\sum_{w\in W_n} x^{\xi w} = (x_0^2x_1\cdots x_n)^\ell \text{ for some integer } \ell$$

From this we immediately infer that there can only be one ξ and that $x^\xi = x^{\xi w}$ for all $w \in W_n$. It is trivial to check that if ξ corresponds to the class of the lattice $[p^{a_1},\ldots,p^{a_n};p^{b_1},\ldots,p^{b_n}]$ with $\mu = a_i+b_i$ for all i, then the monomial $x^\xi = x_0^\mu x_1^{a_1}\cdots x_n^{a_n}$ is fixed by all elements of W_n iff $x^\xi = (x_0^2x_1\cdots x_n)^\ell$ for some integer ℓ. Then $\sum_\xi c(\xi)\eta(\xi)\sum_{w\in W_n} x^{\xi w} = c(\xi)\eta(\xi)|W_n|x^\xi = x^\xi$ and $\xi = p^\ell I_{2n}$. Since $\deg(\Gamma p^\ell I_{2n}\Gamma) = 1$, $\eta(\xi) = |W_n|^{-1}$, so $c(\xi) = 1$ which establishes the result. □

Remark 4.2. *The image under Φ of $\Gamma pI\Gamma$ is (the class of) is $x_0^2x_1\cdots x_n$. For comparison, one should note that for a simultaneous Hecke eigenform of weight k for $\Gamma = \Gamma_n$, the associated Satake p-parameters $(\alpha_0,\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^{n+1}/W_n$ satisfy $\alpha_0^2\alpha_1\cdots\alpha_n = p^{nk-n(n+1)/2}$, and also that under the Satake isomorphism (Andrianov's spherical map Ω), the image of $\Gamma pI\Gamma$ is $p^{-n(n+1)/2}x_0^2x_1\cdots x_n$.*

Corollary 4.3. *The connecting map $\rho : \mathbb{Q}[x_0^{\pm1},\ldots,x_n^{-1}]^{W_n} \to \mathrm{End}(\mathcal{B})$ defined by $\rho(t(\xi)) = \eta(\xi)^{-1}T_{\mathcal{B}}(\xi)$ is an injective ring homomorphism. satisfying $\rho\circ\Phi = \Psi$.*

Proof. Both Φ and Ψ have the same kernel K, so induce injective homomorphisms from $\widetilde{\Phi} : \mathcal{H}_p/K \to \mathbb{Q}[x_0^{\pm1},\ldots,x_n^{-1}]^{W_n}$, and $\widetilde{\Psi} : \mathcal{H}_p/K \to \mathrm{End}(\mathcal{B})$. Since, Φ and hence $\widetilde{\Phi}$ is surjective, $\rho = \widetilde{\Psi}\circ\widetilde{\Phi}^{-1}$ has the desired properties. □

Remark 4.4. *We remark that while closely related, there is a difference between the spherical map Ω defined by Andrianov [1] and our map Φ.*

Let $\xi_0 = \begin{pmatrix} I_2 & 0 \\ 0 & pI_2 \end{pmatrix}$ *and* $\xi_k = \begin{pmatrix} I_{n-k} & 0 & 0 & 0 \\ 0 & pI_k & 0 & 0 \\ 0 & 0 & p^2 I_{n-k} & 0 \\ 0 & 0 & 0 & pI_k \end{pmatrix}$, $1 \le k \le n$. *By Lemma 3.13,*

$$t(\xi_0) = \sum_{w \in W_n} x^{\xi_0 w} = |W_n|[W_n : \mathrm{Stab}(x_0)]^{-1} \sum_{\varepsilon_i = 0,1} x_0 x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} = \frac{|W_n|}{2^n} t_0(p),$$

so that

$$\Phi(T(p)) = \eta(\xi_0) t(\xi_0) = \frac{\deg(\xi_0)}{|W_n|} \frac{|W_n|}{2^n} \sum_{\varepsilon_i = 0,1} x_0 x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} = \frac{\deg(\xi_0)}{2^n} \Omega(T(p)) = \frac{\deg(\xi_0)}{2^n} t_0(p).$$

As mentioned earlier for $1 \le k < n$, $T_{n-k}(p^2)$ corresponds to $t_k(p)$, so we have

$$t(\xi_k) = \sum_{w \in W_n} x^{\xi_k w} = |W_n|[W_n : \mathrm{Stab}(\xi_k)]^{-1} \sum_{1 \le i_1 \le \cdots \le i_k \le n,\ \delta_{i_j} = \pm 1} x_{i_1}^{\delta_{i_1}} \cdots x_{i_k}^{\delta_{i_k}} = \frac{|W_n|}{2^k \binom{n}{k}} t_k(p),$$

so that

$$\Phi(T_{n-k}(p^2)) = \eta(\xi_k) t(\xi_k) = \frac{\deg(\xi_k)}{|W_n|} \frac{|W_n|}{2^k \binom{n}{k}} t_k(p) = \frac{\deg(\xi_k)}{2^k \binom{n}{k}} t_k(p)$$

Computations with the spherical map are far less trivial. For comparison purposes we compute $\Omega(T_1^2(p^2))$. We have already shown that

$$\Phi(T_1^2(p^2)) = \frac{\deg(\xi_1)}{4} t_1(p) = \frac{\deg(\xi_1)}{4} (x_1 + x_1^{-1} + x_2 + x_2^{-1})$$

while

$$\Omega(T_2^2(p^2)) = \Omega(pI_4) = p^{-3} x_0^2 x_1 x_2$$

and

$$\begin{aligned} \Omega(T_1^2(p^2)) &= p^{-1}(x_0^2 x_1 + x_0^2 x_2 + x_0^2 x_1^2 x_2 + x_0^2 x_1 x_2^2) + p^{-2}(p^2 - 1)(x_0^2 x_1 x_2) \\ &= (x_0^2 x_1 x_2)[p^{-1}(x_1 + x_1^{-1} + x_2 + x_2^{-1}) + p^{-2}(p^2 - 1)] \\ &= (x_0^2 x_1 x_2)[p^{-1} t_1(p) + p^{-2}(p^2 - 1)] \\ &= \Omega(T_2^2(p^2))[p^2 t_1(p) + (p^2 - 1)] \end{aligned}$$

So it is clear that the action of Ω is somewhat more complicated, intertwining the Hecke operators in comparison to the action of Φ.

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