Rolle's Theorem over Local Fields

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Abstract

In this paper we show that no non-archimedean local field has Rolle's property.

1 Introduction

Rolle's property for a field K is that if f is a polynomial in $K[x]$ which splits over K, then its derivative splits over K. This property is implied by the usual Rolle's theorem taught in calculus for functions over the real numbers, however for fields with no ordering, it is the best one can hope for. Of course, Rolle's property holds not only for the real numbers, but also for any algebraically- or real-closed field. Kaplansky ([3], p. 30) asks for a characterization of all such fields.

For finite fields, Rolle's property holds only for the fields with 2 and 4 elements [2], [1]. In this paper, we show that Rolle's property fails to hold over any non-archimedean local field (with a finite residue class field). In particular, such fields include the completion of any global field of number theory with respect to a nontrivial non-archimedean valuation. Moreover, we show that there are counterexamples for Rolle's property for polynomials of lowest possible degree, namely cubics.

2 Rolle's Theorem

Theorem 2.1. *Rolle's property fails to hold over any non-archimedean local field having finite residue class field.*

Proof. Let K be a non-archimedean local field with finite residue class field. Let $\mathcal{O}$ be the ring of integers of K, and $\mathcal{P}$ its maximal ideal. Let $k = \mathcal{O}/\mathcal{P} \cong \mathbb{F}_q$ be its residue class field having q elements.

The proof requires a few elementary results from the theory of quadratic forms, which the reader can easily glean from [4]. A critical fact we need (see 63.9 [4]) is that the index of squares $[K^\times : (K^\times)^2] = 4 \cdot q^e \geq 4$ where $2\mathcal{O} = \mathcal{P}^e$.

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Consider the family of polynomials $F = F_{b,c}(x) = x(x-b)(x-c)$ for $b, c \in K^\times$. The derivative of F is $F'(x) = 3x^2 - 2(b+c)x + bc$ which splits over K if and only if its discriminant $\Delta = 4(b^2 - bc + c^2)$ is a square in K. Of course, this is true if and only if $b^2 - bc + c^2$ is a square in K.

Consider the binary quadratic form $Q(x, y) = x^2 - xy + y^2$. If Q is isotropic, it is universal, so in particular it represents all elements of $K^\times$. Since from above $(K^\times)^2 \subsetneq K^\times$, it is clear Q represents a non-square. Choosing b and c to achieve this produces the counterexample.

Finally, suppose Q is anisotropic, and let $Q(K^\times)$ denote the values assumed by Q on $K^\times$. Then since Q obviously represents 1, it is immediate from 63:15 of [4] that $Q(K^\times)$ is a subgroup of index 2 in $K^\times$, and contains $(K^\times)^2$. In brief, $Q \cong < 1, -\alpha >$ for a non-square α, and $Q(K^\times)$ is the kernel of the surjective homomorphism $\gamma \mapsto \left(\frac{\alpha,\gamma}{\mathcal{P}}\right)$ of $K^\times \to \{\pm 1\}$, with $\left(\frac{\alpha,\gamma}{\mathcal{P}}\right)$ the Hilbert symbol. But as noted above, $[K^\times : (K^\times)^2] \geq 4$, so once again Q represents a non-square in K, and we are done.

□

Remark 2.2. *We note that in the case $q \neq 2, 4$, the result above could also be deduced from the corresponding result over finite fields. By [2],[1], Rolle's property fails to hold over the (finite) residue class field k, so let $f \in k[x]$ be a counterexample to Rolle's property, that is a polynomial which splits over k, but whose derivative does not. We may write $f(x) = (x - \bar{a}_1) \cdots (x - \bar{a}_r)$ for elements $a_i \in \mathcal{O}$. Consider the polynomial $F(x) = (x - a_1) \cdots (x - a_r) \in \mathcal{O}[x]$. By definition, F splits over K. However its derivative cannot, since the reduction of F' mod $\mathcal{P}$ agrees with the derivative of f. This produces a counterexample in these cases.*

References

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