

Other supplemental material

1. The $\mathcal{O}(\varepsilon^3)$ coefficients

Here, we give the full expressions for third-order coefficients, $\text{KIN}_{3,m}$ and $\text{DYN}_{3,m}$, defined in (A 1) and (A 2) as the coefficients of the lower-order terms' m -th harmonics in the kinematic and dynamic boundary conditions, respectively. Recall that $\hat{P}_m$ was defined in (2.17) as the pressure's Fourier coefficient multiplying the m -th harmonic of the wave profile: $\hat{p}_m(t) = k\hat{P}_m\hat{\eta}_m(t)$.

$$\begin{aligned}
 \text{KIN}_{3,1} = & \frac{1}{4} \left(\text{i} \left(\left(\left(\frac{3}{4} + \hat{P}_1 - \frac{\hat{P}_2}{4} \right) \cosh^4(h) + \left(-\frac{1}{4} - \frac{\hat{P}_1}{2} + \frac{\hat{P}_2}{4} \right) \cosh^2(h) + \frac{\hat{P}_1}{4} + \frac{1}{4} \right) \left| \hat{P}_1 + 1 \right| \right. \right. \\
 & + \frac{1}{2} \left(\left(\hat{P}_2 + 1 \right) \cosh^4(h) + \left(\hat{P}_1 + 1 \right) \cosh^2(h) - \frac{\hat{P}_1}{2} - \frac{1}{2} \right) \left(\hat{P}_1 + 1 \right) \Bigg) \\
 & \times \left(\left(\left(\frac{1}{2} + \hat{P}_1 - \frac{\hat{P}_2}{2} \right) \cosh^2(h) - \frac{\hat{P}_1}{2} - \frac{1}{2} \right) \omega \sinh(h) \cosh(h) \right)^{-1}
 \end{aligned} \tag{1.1}$$

$$\text{KIN}_{3,3} = \frac{3}{8} \frac{\text{i} \left(3 \cosh^2(h) - 1 \right) \left(\cosh^2(h) \hat{P}_2 + \cosh^2(h) + 2\hat{P}_1 + 2 \right) \omega}{\left(2 \cosh^2(h) \hat{P}_1 - \cosh^2(h) \hat{P}_2 + \cosh^2(h) - \hat{P}_1 - 1 \right) \sinh^2(h)} \tag{1.2}$$

$$\begin{aligned}
\text{DYN}_{3,1} = & - \left(\left((\hat{P}_1 + 1) \cosh^4(h) + \left(-\frac{3}{4}\hat{P}_1 + \frac{\hat{P}_2}{4} - \frac{1}{2} \right) \cosh^2(h) - \frac{\hat{P}_1}{8} - \frac{1}{8} \right) \hat{P}_1 |\hat{P}_1 + 1| \right. \\
& + \frac{1}{2} \left(\left(\left(-\hat{P}_1 + \frac{\hat{P}_2}{4} - \frac{3}{4} \right) |\hat{P}_1|^2 + \left(\hat{P}_1^2 + \left(-\frac{3}{2}\hat{P}_2 - \frac{1}{2} \right) \hat{P}_1 - \frac{5}{4}\hat{P}_2 - \frac{5}{4} \right) \hat{P}_1 \right) \cosh^2(h) - \frac{5}{2} (\hat{P}_1 + 1)^2 \hat{P}_1 \right) (\cosh(h) + 1) (\cosh(h) - 1) \\
& \times \left(\sinh^2(h) \hat{P}_1 (2 \cosh^2(h) \hat{P}_1 - \cosh^2(h) \hat{P}_2 + \cosh^2(h) - \hat{P}_1 - 1) \right)^{-1} \quad (1.3)
\end{aligned}$$

$$\text{DYN}_{3,3} = -\frac{1}{8} \frac{\left(4 \cosh^4(h) \hat{P}_1 - 7 \cosh^4(h) \hat{P}_2 - 3 \cosh^4(h) - 18 \cosh^2(h) \hat{P}_1 + 9 \cosh^2(h) \hat{P}_2 - 9 \cosh^2(h) + 15 \hat{P}_1 + 15 \right) (\hat{P}_1 + 1)}{\left(2 \cosh^2(h) \hat{P}_1 - \cosh^2(h) \hat{P}_2 + \cosh^2(h) - \hat{P}_1 - 1 \right) \sinh^2(h)} \quad (1.4)$$

Recall that $\text{COMB}_{3,1}$ is defined in (A 15) as the combination of $\text{KIN}_{3,1}$ and $\text{DYN}_{3,1}$ formed by eliminating $\hat{\phi}_{3,1}$. This grouping appears often in the 4th-order coefficients and governs the higher-order corrections to the phase speed (A 65) and growth rate (A 66). Therefore, we give its expression here.

$$\begin{aligned}
\text{COMB}_{3,1} = & \quad (1.5) \\
& \frac{1}{6} (\hat{P}_1 + 1) \left(\frac{1}{2} \hat{P}_1 (8 \cosh^4(h) \hat{P}_1 - 2 \cosh^4(h) \hat{P}_2 + 6 \cosh^4(h) - 6 \cosh^2(h) \hat{P}_1 + 2 \cosh^2(h) \hat{P}_2 - 4 \cosh^2(h) + \hat{P}_1 + 1) |\hat{P}_1 + 1| \right. \\
& + \frac{1}{4} \left((-8 \hat{P}_1 + 2 \hat{P}_2 - 6) |\hat{P}_1|^2 + 4 \hat{P}_1^3 - 2 \hat{P}_1^2 \hat{P}_2 - 2 \hat{P}_1^2 - 4 \hat{P}_1 \right) \cosh^4(h) - \frac{5}{2} \left(\left(-\frac{2}{5} - \frac{3}{5} \hat{P}_1 + \frac{\hat{P}_2}{5} \right) |\hat{P}_1|^2 + \left(\hat{P}_1 \left(\hat{P}_1 - \frac{3}{5} \hat{P}_2 \right) + \frac{4}{5} \hat{P}_1 \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{2}{5}\hat{P}_2\Big)\hat{P}_1\Big)\cosh^2(h)+2\Big(\hat{P}_1^2-\frac{1}{8}|\hat{P}_1|^2+\frac{7\hat{P}_1}{8}\Big)(\hat{P}_1+1)\Big)\Bigg)\Bigg(\Bigg(\Big((1-\hat{P}_2+2\hat{P}_1)\cosh^2(h)^2-\hat{P}_1-1\Big)\cosh(h)\hat{P}_1\omega\Big(\hat{P}_1-\frac{1}{3}|\hat{P}_1+1| \\
& +1\Big)\sinh(h)\Big)^{-1}
\end{aligned} \tag{1.6}$$

2. The $\mathcal{O}(\varepsilon^4)$ coefficients

Here, we give the full expressions for fourth-order coefficients, $\text{KIN}_{4,m}$ and $\text{DYN}_{4,m}$, defined in (A 28) and (A 29) as the coefficients of the lower-order terms' m -th harmonics in the kinematic and dynamic boundary conditions, respectively.

$\text{KIN}_{4,2} =$

$$\begin{aligned}
& \frac{1}{2}(\hat{P}_1+1)\Bigg(-2\Big(\cosh^2(h)+\frac{1}{2}\Big)\Bigg(\frac{1}{2}\text{i}\sinh(h)\text{COMB}_{3,1}\Bigg(\Bigg(\frac{2}{3}+\hat{P}_1-\frac{\hat{P}_3}{3}\Bigg)\cosh^2(h)-\frac{2}{3}-\frac{3}{4}\hat{P}_1+\frac{\hat{P}_3}{12}\Bigg)\cosh(h)\omega+\text{i}\Bigg(\Bigg(\frac{2}{3}+\hat{P}_1 \\
& -\frac{\hat{P}_3}{3}\Bigg)\Bigg(\frac{9}{8}+\hat{P}_1+\frac{\hat{P}_2}{8}\Bigg)\cosh^4(h)-\frac{(13\hat{P}_1+13)\cosh^2(h)}{8}\Bigg(\frac{29}{39}+\hat{P}_1-\frac{10\hat{P}_3}{39}\Bigg)+\frac{3}{4}(\hat{P}_1+1)\Bigg(\frac{31}{36}+\hat{P}_1-\frac{5\hat{P}_3}{36}\Bigg)\Bigg)\Bigg)\hat{P}_1|\hat{P}_1+1| \\
& +3\text{i}\sinh(h)(\hat{P}_1+1)\Big(\cosh^2(h)+\frac{1}{2}\Big)\text{COMB}_{3,1}\hat{P}_1\Bigg(\Bigg(\frac{2}{3}+\hat{P}_1-\frac{\hat{P}_3}{3}\Bigg)\cosh^2(h)-\frac{2}{3}-\frac{3}{4}\hat{P}_1+\frac{\hat{P}_3}{12}\Bigg)\cosh(h)\omega+\text{i}\Bigg(\Bigg(\frac{25}{36}+\hat{P}_1^2 \\
& +\Bigg(-\frac{5\hat{P}_3}{18}-\frac{\hat{P}_2}{12}\Bigg)\hat{P}_1+\frac{1}{18}\hat{P}_2\hat{P}_3+\frac{59\hat{P}_1}{36}-\frac{2}{9}\hat{P}_3-\frac{\hat{P}_2}{36}\Bigg)|\hat{P}_1|^2-\frac{1}{2}\Bigg(-\frac{5}{9}+\Bigg(\hat{P}_1^2+\Bigg(\frac{2}{3}\hat{P}_3-2\hat{P}_2\Bigg)\hat{P}_1+\frac{7}{6}\hat{P}_2\hat{P}_3\Bigg)\hat{P}_1-\frac{1}{3}\hat{P}_1^2
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{55\hat{P}_3}{18} - \frac{8}{3}\hat{P}_2 \right) \hat{P}_1 + \frac{19\hat{P}_2\hat{P}_3}{18} - \frac{16\hat{P}_1}{9} + \frac{41\hat{P}_3}{18} - \frac{7\hat{P}_2}{9} \Big) \hat{P}_1 \Big) \cosh^6(h) + \left(-\frac{17|\hat{P}_1|^2}{12} \left(\frac{10}{17} + \hat{P}_1^2 + \left(-\frac{23\hat{P}_3}{102} - \frac{4\hat{P}_2}{17} \right) \hat{P}_1 + \frac{5\hat{P}_2\hat{P}_3}{102} \right. \right. \\
& + \frac{157\hat{P}_1}{102} - \frac{3\hat{P}_3}{17} - \frac{19\hat{P}_2}{102} \Big) + \frac{25\hat{P}_1}{8} \left(-\frac{8}{75} + \hat{P}_1 \left(\hat{P}_1^2 + \left(-\frac{18\hat{P}_2}{25} - \frac{4\hat{P}_3}{15} \right) \hat{P}_1 + \frac{11\hat{P}_2\hat{P}_3}{75} \right) + \frac{39\hat{P}_1^2}{25} + \left(-\frac{64\hat{P}_3}{225} - \frac{89\hat{P}_2}{75} \right) \hat{P}_1 + \frac{28\hat{P}_2\hat{P}_3}{225} \right. \\
& + \frac{107\hat{P}_1}{225} - \frac{\hat{P}_3}{25} - \frac{22\hat{P}_2}{45} \Big) \Big) \cosh^4(h) + \left(-\frac{23|\hat{P}_1|^2}{48} \left(\frac{16}{23} + \hat{P}_1^2 + \left(-\frac{5\hat{P}_3}{69} - \frac{6\hat{P}_2}{23} \right) \hat{P}_1 + \frac{2\hat{P}_2\hat{P}_3}{69} + \frac{5}{3}\hat{P}_1 - \frac{\hat{P}_3}{23} - \frac{16\hat{P}_2}{69} \right) - \frac{73\hat{P}_1}{16} \left(\frac{104}{219} \right. \right. \\
& + \left(\hat{P}_1^2 + \left(-\frac{18\hat{P}_2}{73} - \frac{17\hat{P}_3}{73} \right) \hat{P}_1 + \frac{2\hat{P}_2\hat{P}_3}{73} \right) \hat{P}_1 + \frac{529\hat{P}_1^2}{219} + \left(-\frac{283\hat{P}_3}{657} - \frac{32\hat{P}_2}{73} \right) \hat{P}_1 + \frac{16\hat{P}_2\hat{P}_3}{657} + \frac{1244\hat{P}_1}{657} - \frac{44\hat{P}_3}{219} - \frac{128\hat{P}_2}{657} \Big) \Big) \cosh^2(h) \\
& + \frac{3\hat{P}_1 + 3}{32} \left(\left(\left(\frac{25}{27} + \hat{P}_1 - \frac{2\hat{P}_3}{27} \right) |\hat{P}_1|^2 + 20 \left(\frac{239}{270} + \hat{P}_1 \left(\hat{P}_1 - \frac{29\hat{P}_3}{180} \right) + \frac{17\hat{P}_1}{9} - \frac{89\hat{P}_3}{540} \right) \hat{P}_1 \right) \right) \Big) \Big) \\
& \times \left(\omega \sinh^2(h) \left(\left(\frac{1}{2} + \hat{P}_1 - \frac{\hat{P}_2}{2} \right) \cosh^2(h) - \frac{\hat{P}_1}{2} - \frac{1}{2} \right) \hat{P}_1 \left(-3\hat{P}_1 + |\hat{P}_1 + 1| - 3 \right) \left(\left(\frac{2}{3} + \hat{P}_1 - \frac{\hat{P}_3}{3} \right) \cosh^2(h) - \frac{2}{3} - \frac{3}{4}\hat{P}_1 + \frac{\hat{P}_3}{12} \right) \right)^{-1}
\end{aligned} \tag{2.1}$$

$$\text{KIN}_{4,4} =$$

$$\frac{1}{4} (\text{i} \cosh(h) \omega) \left(\left(\frac{8}{9} + \hat{P}_1^3 + \left(-\frac{4}{3}\hat{P}_2 + \frac{5}{9}\hat{P}_3 \right) \hat{P}_1^2 + \frac{2}{3} \left(\frac{2}{3}\hat{P}_3 + \hat{P}_2 \right) \hat{P}_2\hat{P}_1 - \frac{4}{9}\hat{P}_2^2\hat{P}_3 + \frac{20\hat{P}_1^2}{9} + \left(-\frac{8\hat{P}_2}{9} + \frac{14\hat{P}_3}{9} \right) \hat{P}_1 + \frac{2}{9}\hat{P}_2 \left(-2\hat{P}_3 + \hat{P}_2 \right) \right) \right)$$

$$\begin{aligned}
& + \frac{23\hat{P}_1}{9} - \frac{4}{9}\hat{P}_2 + \frac{5}{9}\hat{P}_3 \Big) \cosh^8(h) + \left(-\frac{5}{9} - \frac{59\hat{P}_1^3}{12} + \left(\frac{85\hat{P}_2}{12} + \frac{11\hat{P}_3}{12} \right) \hat{P}_1^2 - \frac{49\hat{P}_2\hat{P}_1}{24} \left(\frac{146\hat{P}_3}{147} + \hat{P}_2 \right) + \frac{31\hat{P}_2^2\hat{P}_3}{72} - \frac{27\hat{P}_1^2}{4} \right. \\
& + \left(\frac{145\hat{P}_2}{18} - \frac{7\hat{P}_3}{36} \right) \hat{P}_1 - \frac{29\hat{P}_2}{18} \left(\hat{P}_2 + \frac{21\hat{P}_3}{29} \right) - \frac{203\hat{P}_1}{72} + \frac{11\hat{P}_2}{6} - \frac{49\hat{P}_3}{72} \Big) \cosh^6(h) + \left(\frac{11}{6} + \frac{81\hat{P}_1^3}{8} + \left(-\frac{33\hat{P}_2}{4} - \frac{185\hat{P}_3}{72} \right) \hat{P}_1^2 \right. \\
& + \hat{P}_2 \left(\hat{P}_2 + \frac{59\hat{P}_3}{36} \right) \hat{P}_1 - 1/9\hat{P}_2^2\hat{P}_3 + \frac{176\hat{P}_1^2}{9} + \left(-\frac{463\hat{P}_2}{36} - 7/2\hat{P}_3 \right) \hat{P}_1 + \frac{8\hat{P}_2}{9} \left(\hat{P}_2 + \frac{51\hat{P}_3}{32} \right) + \frac{91\hat{P}_1}{8} - \frac{29\hat{P}_2}{6} - \frac{25\hat{P}_3}{24} \Big) \cosh^4(h) \\
& - \frac{(361\hat{P}_1 + 361) \cosh^2(h)^2}{48} \left(\frac{530}{1083} + \hat{P}_1^2 + \left(-\frac{129\hat{P}_2}{361} - \frac{73\hat{P}_3}{361} \right) \hat{P}_1 + \frac{53\hat{P}_2\hat{P}_3}{1083} + \frac{520\hat{P}_1}{361} - \frac{334\hat{P}_2}{1083} - \frac{166\hat{P}_3}{1083} \right) + \frac{57(\hat{P}_1 + 1)^2}{32} \left(\frac{436}{513} \right. \\
& \left. \left. + \hat{P}_1 - \frac{77\hat{P}_3}{513} \right) \right) \left(\sinh^3(h) \left(\left(\frac{2}{3} + \hat{P}_1 - \frac{\hat{P}_3}{3} \right) \cosh^2(h) - \frac{2}{3} - \frac{3}{4}\hat{P}_1 + \frac{\hat{P}_3}{12} \right) \left(\left(\frac{1}{2} + \hat{P}_1 - \frac{\hat{P}_2}{2} \right) \cosh^2(h) - \frac{\hat{P}_1}{2} - \frac{1}{2} \right)^2 \right)^{-1} \quad (2.2)
\end{aligned}$$

$$\text{KIN}_{4,0} = 0 \quad (2.3)$$

$$\text{DYN}_{4,2} =$$

$$- \frac{1}{4} \left(\left(-\text{COMB}_{3,1} \sinh(h) \left((\hat{P}_2 + 1) \cosh^4(h) + (\hat{P}_1 + 1) \cosh^2(h) - \frac{\hat{P}_1}{2} - \frac{1}{2} \right) \hat{P}_1 \left(\left(\frac{2}{3} + \hat{P}_1 - \frac{\hat{P}_3}{3} \right) \cosh^2(h) - \frac{2}{3} - \frac{3}{4}\hat{P}_1 \right. \right. \right.$$

$$\begin{aligned}
& + \frac{\hat{P}_3}{12} \cosh(h) \omega - \frac{1}{6} \left(\left(\frac{7}{6} + \left(\hat{P}_2 + \frac{\hat{P}_3}{3} \right) \hat{P}_1 - \frac{1}{6} \hat{P}_2 \hat{P}_3 + \frac{4}{3} \hat{P}_1 + \frac{5}{6} \hat{P}_2 + \frac{\hat{P}_3}{6} \right) |\hat{P}_1|^2 + 15 \hat{P}_1 \left(\frac{59}{45} + \hat{P}_1 \left(\hat{P}_1^2 + \left(-\frac{\hat{P}_2}{2} + \frac{4\hat{P}_3}{15} \right) \hat{P}_1 + \frac{7\hat{P}_2 \hat{P}_3}{15} \right) \right. \right. \\
& + \frac{83\hat{P}_1^2}{30} + \left(-\frac{7\hat{P}_2}{15} + \frac{46\hat{P}_3}{45} \right) \hat{P}_1 + \frac{41\hat{P}_2 \hat{P}_3}{90} + \frac{\hat{P}_2}{45} + \frac{67\hat{P}_3}{90} + \frac{139\hat{P}_1}{45} \left. \right) \cosh^8(h) - \frac{1}{6} \left(\frac{7}{2} \left(\frac{5}{21} + \hat{P}_1^2 + \left(-\frac{13\hat{P}_2}{28} - \frac{17\hat{P}_3}{42} \right) \hat{P}_1 + \frac{3\hat{P}_2 \hat{P}_3}{28} \right. \right. \\
& + \frac{95\hat{P}_1}{84} - \frac{5\hat{P}_2}{14} - \frac{25\hat{P}_3}{84} \left. \right) |\hat{P}_1|^2 - \frac{165\hat{P}_1}{4} \left(\frac{2}{9} + \hat{P}_1 \left(\hat{P}_1^2 + \left(-\frac{8\hat{P}_3}{55} - \frac{93\hat{P}_2}{110} \right) \hat{P}_1 + \frac{7\hat{P}_2 \hat{P}_3}{30} \right) + \frac{127\hat{P}_1^2}{66} + \left(-\frac{78\hat{P}_2}{55} - \frac{23\hat{P}_3}{990} \right) \hat{P}_1 + \frac{37\hat{P}_2 \hat{P}_3}{165} \right. \\
& + \frac{52\hat{P}_1}{45} - \frac{32\hat{P}_2}{55} + \frac{56\hat{P}_3}{495} \left. \right) \cosh^6(h) - \frac{1}{6} \left(-\frac{11}{2} \left(\frac{7}{11} + \hat{P}_1^2 + \left(-\frac{2}{11} \hat{P}_2 - \frac{5\hat{P}_3}{22} \right) \hat{P}_1 + \frac{1}{22} \hat{P}_2 \hat{P}_3 + \frac{35\hat{P}_1}{22} - \frac{3\hat{P}_2}{22} - \frac{2}{11} \hat{P}_3 \right) |\hat{P}_1|^2 + 63\hat{P}_1 \left(\frac{2}{7} \right. \right. \\
& + \left(\hat{P}_1^2 + \left(-\frac{16\hat{P}_3}{63} - \frac{13\hat{P}_2}{28} \right) \hat{P}_1 + \frac{5\hat{P}_2 \hat{P}_3}{84} \right) \hat{P}_1 + \frac{79\hat{P}_1^2}{36} + \left(-\frac{215\hat{P}_2}{252} - \frac{3}{7} \hat{P}_3 \right) \hat{P}_1 + \frac{1}{18} \hat{P}_2 \hat{P}_3 + \frac{187\hat{P}_1}{126} - \frac{11\hat{P}_2}{28} - \frac{5\hat{P}_3}{28} \left. \right) \cosh^4(h) \\
& - \frac{1}{6} \left(2 \left(\frac{73}{96} + \hat{P}_1^2 + \left(-\frac{3}{16} \hat{P}_2 - \frac{7\hat{P}_3}{96} \right) \hat{P}_1 + \frac{1}{48} \hat{P}_2 \hat{P}_3 + \frac{167\hat{P}_1}{96} - \frac{\hat{P}_2}{6} - \frac{5\hat{P}_3}{96} \right) |\hat{P}_1|^2 - \frac{747\hat{P}_1}{16} \left(\frac{1538}{2241} + \left(\hat{P}_1^2 + \left(-\frac{18\hat{P}_3}{83} - \frac{6\hat{P}_2}{83} \right) \hat{P}_1 \right. \right. \\
& + \frac{2\hat{P}_2 \hat{P}_3}{249} \left. \right) \hat{P}_1 + \frac{1993\hat{P}_1^2}{747} + \left(-\frac{32\hat{P}_2}{249} - \frac{947\hat{P}_3}{2241} \right) \hat{P}_1 + \frac{16\hat{P}_2 \hat{P}_3}{2241} + \frac{5278\hat{P}_1}{2241} - \frac{128\hat{P}_2}{2241} - \frac{463\hat{P}_3}{2241} \left. \right) \cosh^2(h) + \frac{1}{288} \left(\left(\hat{P}_3 + 1 \right) |\hat{P}_1|^2 - 486\hat{P}_1^3 \right)
\end{aligned}$$

$$\begin{aligned}
& + 72\hat{P}_1^2\hat{P}_3 - 900\hat{P}_1^2 + 73\hat{P}_1\hat{P}_3 - 413\hat{P}_1 \Big) \Big(\hat{P}_1 + 1 \Big) \Big) \Big| \hat{P}_1 + 1 \Big| - 3 \Big(\hat{P}_1 + 1 \Big) \left(-\text{COMB}_{3,1} \sinh(h) \left(\left(\hat{P}_2 + 1 \right) \cosh^4(h) + \left(\hat{P}_1 + 1 \right) \cosh^2(h) \right. \right. \\
& \left. \left. - \frac{\hat{P}_1}{2} - \frac{1}{2} \right) \hat{P}_1 \left(\left(\frac{2}{3} + \hat{P}_1 - \frac{\hat{P}_3}{3} \right) \cosh^2(h) - \frac{2}{3} - \frac{3}{4}\hat{P}_1 + \frac{\hat{P}_3}{12} \right) \cosh(h) \omega - \frac{1}{3} \left(\left(\frac{3}{2} + \hat{P}_1 + \frac{\hat{P}_2}{2} \right) \left(\frac{7}{6} + \hat{P}_1 + \frac{\hat{P}_3}{6} \right) \Big| \hat{P}_1 \Big|^2 + \left(\frac{11}{6} + \hat{P}_1 \left(\hat{P}_1^2 \right. \right. \right. \\
& \left. \left. + \left(\frac{7}{6}\hat{P}_3 - \frac{17\hat{P}_2}{4} \right) \hat{P}_1 + \frac{13\hat{P}_2\hat{P}_3}{6} \right) + \frac{11\hat{P}_1^2}{12} + \left(-\frac{35\hat{P}_2}{6} + \frac{14}{3}\hat{P}_3 \right) \hat{P}_1 + \frac{9}{4}\hat{P}_2\hat{P}_3 - \frac{3}{2}\hat{P}_2 + \frac{43\hat{P}_3}{12} + \frac{5}{3}\hat{P}_1 \right) \hat{P}_1 \right) \cosh^8(h) - \frac{1}{3} \left(\frac{1}{2} \left(-\frac{2}{3} + \hat{P}_1^2 \right. \right. \\
& \left. \left. + \left(-\frac{5}{4}\hat{P}_3 - \frac{\hat{P}_2}{2} \right) \hat{P}_1 + \frac{1}{12}\hat{P}_2\hat{P}_3 + \frac{\hat{P}_1}{4} - \frac{5\hat{P}_2}{12} - \frac{7}{6}\hat{P}_3 \right) \Big| \hat{P}_1 \Big|^2 - \frac{49\hat{P}_1}{4} \left(\frac{1}{21} + \left(\hat{P}_1^2 + \left(-\frac{67\hat{P}_3}{294} - \frac{51\hat{P}_2}{49} \right) \hat{P}_1 + \frac{85\hat{P}_2\hat{P}_3}{294} \right) \hat{P}_1 + \frac{71\hat{P}_1^2}{42} \right. \right. \\
& \left. \left. + \left(-\frac{521\hat{P}_2}{294} - \frac{17\hat{P}_3}{147} \right) \hat{P}_1 + \frac{2}{7}\hat{P}_2\hat{P}_3 + \frac{109\hat{P}_1}{147} - \frac{36\hat{P}_2}{49} + \frac{16\hat{P}_3}{147} \right) \right) \cosh^6(h) - \frac{1}{3} \left(-\frac{35\Big| \hat{P}_1 \Big|^2}{16} \left(\frac{4}{5} + \hat{P}_1^2 + \frac{4\hat{P}_2\hat{P}_3}{105} - \frac{5\hat{P}_1\hat{P}_3}{21} + \frac{37\hat{P}_1}{21} + \frac{4\hat{P}_2}{105} \right. \right. \\
& \left. \left. - \frac{\hat{P}_3}{5} \right) + \frac{425\hat{P}_1}{16} \left(\frac{148}{425} + \hat{P}_1 \left(\hat{P}_1^2 + \left(-\frac{111\hat{P}_3}{425} - \frac{162\hat{P}_2}{425} \right) \hat{P}_1 + \frac{24\hat{P}_2\hat{P}_3}{425} \right) + \frac{967\hat{P}_1^2}{425} + \left(-\frac{12\hat{P}_2}{17} - \frac{569\hat{P}_3}{1275} \right) \hat{P}_1 + \frac{4\hat{P}_2\hat{P}_3}{75} + \frac{122\hat{P}_1}{75} - \frac{418\hat{P}_2}{1275} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{16\hat{P}_3}{85}\Bigg)\Bigg)\cosh^4(h)-\frac{1}{3}\Bigg(\frac{17|\hat{P}_1|^2}{32}\Bigg(\frac{15}{17}+\hat{P}_1+\Bigg(\frac{10\hat{P}_3}{51}-\frac{6\hat{P}_2}{17}\Bigg)\hat{P}_1+\frac{2\hat{P}_2\hat{P}_3}{51}+\frac{94\hat{P}_1}{51}-\frac{16\hat{P}_2}{51}+\frac{4\hat{P}_3}{17}\Bigg)-\frac{597\hat{P}_1}{32}\Bigg(\frac{1274}{1791} \\
& +\hat{P}_1\Bigg(\hat{P}_1^2+\Bigg(-\frac{328\hat{P}_3}{1791}-\frac{18\hat{P}_2}{199}\Bigg)\hat{P}_1+\frac{2\hat{P}_2\hat{P}_3}{199}\Bigg)+\frac{4832\hat{P}_1^2}{1791}+\Bigg(-\frac{32\hat{P}_2}{199}-\frac{72\hat{P}_3}{199}\Bigg)\hat{P}_1+\frac{16\hat{P}_2\hat{P}_3}{1791}+\frac{1439\hat{P}_1}{597}-\frac{128\hat{P}_2}{1791}-\frac{322\hat{P}_3}{1791}\Bigg)\Bigg)\cosh^2(h) \\
& -\frac{1}{16}\Bigg(\Bigg(\frac{7}{9}+\hat{P}_1-\frac{2}{9}\hat{P}_3\Bigg)|\hat{P}_1|^2+18\hat{P}_1\Bigg(\frac{293}{324}+\hat{P}_1\Bigg(\hat{P}_1-\frac{5\hat{P}_3}{36}\Bigg)+\frac{23\hat{P}_1}{12}-\frac{49\hat{P}_3}{324}\Bigg)\Bigg)(\hat{P}_1+1)\Bigg)\Bigg)\Bigg(\Big(\cosh(h)+1\Big)\sinh(h)\Big(-3\hat{P}_1 \\
& +|\hat{P}_1+1|-3\Big)\Bigg(\Big(\frac{1}{2}+\hat{P}_1-\frac{\hat{P}_2}{2}\Big)\cosh^2(h)-\frac{\hat{P}_1}{2}-\frac{1}{2}\Big)(\cosh(h)-1)\hat{P}_1\Bigg(\Big(\frac{2}{3}+\hat{P}_1-\frac{\hat{P}_3}{3}\Big)\cosh^2(h)-\frac{2}{3}-\frac{3}{4}\hat{P}_1+\frac{\hat{P}_3}{12}\Big)\cosh(h)\Bigg)^{-1} \\
& \hspace{15cm} (2.4)
\end{aligned}$$

DYN_{4,4} =

$$\begin{aligned}
& \frac{1}{8}(\hat{P}_1+1)\Bigg(\Bigg(\frac{4}{9}+\hat{P}_1^3+\Bigg(\frac{7\hat{P}_3}{9}-\frac{19\hat{P}_2}{6}\Bigg)\hat{P}_1^2+\frac{\hat{P}_2\hat{P}_1}{12}\Bigg(\frac{38\hat{P}_3}{3}+19\hat{P}_2\Bigg)-\frac{29\hat{P}_2^2\hat{P}_3}{36}+\frac{11\hat{P}_1^2}{18}+\Bigg(-\frac{19\hat{P}_2}{9}+\frac{47\hat{P}_3}{18}\Bigg)\hat{P}_1+\frac{7\hat{P}_2}{9}\Bigg(-\frac{5}{7}\hat{P}_3+\hat{P}_2\Big) \\
& -\frac{5}{9}\hat{P}_2+\frac{37\hat{P}_3}{36}+\frac{31\hat{P}_1}{36}\Bigg)\cosh^{10}(h)+\Bigg(-\frac{5}{9}-\frac{115\hat{P}_1^3}{12}+\Bigg(\frac{46\hat{P}_2}{3}+\frac{53\hat{P}_3}{36}\Bigg)\hat{P}_1^2-\frac{14}{3}\hat{P}_2\Bigg(\hat{P}_2+\frac{20\hat{P}_3}{21}\Bigg)\hat{P}_1+\frac{4}{3}\hat{P}_2^2\hat{P}_3-\frac{215\hat{P}_1^2}{18}+\Bigg(\frac{152\hat{P}_2}{9}
\end{aligned}$$

$$\begin{aligned}
& -\frac{3}{2}\hat{P}_3\Big)\hat{P}_1 - \frac{10}{3}\hat{P}_2\left(\frac{8\hat{P}_3}{15} + \hat{P}_2\right) - \frac{17\hat{P}_1}{4} + \frac{38\hat{P}_2}{9} - \frac{59\hat{P}_3}{36}\Big)\cosh^8(h) + \left(\frac{145}{36} + \frac{581\hat{P}_1^3}{24} + \left(-\frac{1067\hat{P}_2}{48} - \frac{439\hat{P}_3}{72}\right)\hat{P}_1^2 + \frac{347\hat{P}_2\hat{P}_1}{96}\left(\hat{P}_2\right.\right. \\
& \left. + \frac{482\hat{P}_3}{347}\right) - \frac{47\hat{P}_2^2\hat{P}_3}{96} + \frac{6379\hat{P}_1^2}{144} + \left(-\frac{773\hat{P}_2}{24} - \frac{1033\hat{P}_3}{144}\right)\hat{P}_1 + \frac{25\hat{P}_2}{8}\left(\frac{97\hat{P}_3}{75} + \hat{P}_2\right) + \frac{7087\hat{P}_1}{288} - \frac{263\hat{P}_2}{24} - \frac{451\hat{P}_3}{288}\Big)\cosh^6(h) \\
& + \left(-\frac{671}{72} - \frac{2257\hat{P}_1^3}{96} + \left(\frac{363\hat{P}_2}{32} + \frac{1447\hat{P}_3}{288}\right)\hat{P}_1^2 - \frac{1}{2}\hat{P}_2\left(\hat{P}_2 + \frac{499\hat{P}_3}{144}\right)\hat{P}_1 + \frac{1}{18}\hat{P}_2^2\hat{P}_3 - \frac{15599\hat{P}_1^2}{288} + \left(\frac{5747\hat{P}_2}{288} + \frac{2395\hat{P}_3}{288}\right)\hat{P}_1\right. \\
& \left. - \frac{4}{9}\hat{P}_2\left(\frac{467\hat{P}_3}{128} + \hat{P}_2\right) - \frac{1441\hat{P}_1}{36} + \frac{157\hat{P}_2}{18} + \frac{241\hat{P}_3}{72}\right)\cosh^4(h) + \frac{(26\hat{P}_1 + 26)\cosh^2(h)}{3}\left(\frac{1753}{2496} + \hat{P}_1^2 + \left(-\frac{129\hat{P}_2}{832} - \frac{103\hat{P}_3}{624}\right)\hat{P}_1\right. \\
& \left. + \frac{7\hat{P}_2\hat{P}_3}{312} + \frac{4193\hat{P}_1}{2496} - \frac{331\hat{P}_2}{2496} - \frac{89\hat{P}_3}{624}\right) - \frac{105(\hat{P}_1 + 1)^2}{128}\left(\frac{788}{945} + \hat{P}_1 - \frac{157\hat{P}_3}{945}\right)\Big) \\
& \times \left(\cosh(h)\left(\left(\frac{2}{3} + \hat{P}_1 - \frac{\hat{P}_3}{3}\right)\cosh^2(h) - \frac{2}{3} - \frac{3}{4}\hat{P}_1 + \frac{\hat{P}_3}{12}\right)\left(\left(\frac{1}{2} + \hat{P}_1 - \frac{\hat{P}_2}{2}\right)\cosh^2(h) - \frac{\hat{P}_1}{2} - \frac{1}{2}\right)^2\sinh^3(h)\right)^{-1}
\end{aligned} \tag{2.5}$$

DYN_{4,0} =

$$\frac{i}{\text{COMB}_{3,1}} \left(\left(\left(-\frac{1}{6} \left(13i\hat{P}_1 \cosh(h) \left(\left(\hat{P}_1 - \frac{\hat{P}_2}{2} + \frac{1}{2} \right) \cosh^2(h) - \frac{\hat{P}_1}{2} - \frac{1}{2} \right) \sinh(h) \left(\left(2|\hat{P}_1|^2\hat{P}_2 + \hat{P}_1 \left(-|\hat{P}_2|^2 + \hat{P}_2 \right) \right) \cosh^2(h) \right. \right. \right. \right. \right. \right.$$

$$\begin{aligned}
& \times -\hat{P}_2 \left(\left| \hat{P}_1 \right|^2 + \hat{P}_1 \right) \omega \left(\left(-\frac{3}{13} \text{COMB}_{3,1}^2 \left| \hat{P}_1 \right|^4 + \hat{P}_1 \left(\left(\hat{P}_1 + 1 \right) \left| \text{COMB}_{3,1} \right|^2 - \text{COMB}_{3,1}^2 \left(\hat{P}_1 + \frac{19}{13} \right) \right) \left| \hat{P}_1 \right|^2 + \hat{P}_1^2 \left(\left(\frac{3}{13} \hat{P}_1^2 + \frac{19\hat{P}_1}{13} \right. \right. \right. \\
& \left. \left. \left. + \frac{16}{13} \right) \left| \text{COMB}_{3,1} \right|^2 - \text{COMB}_{3,1}^2 \left(\hat{P}_1 + \frac{16}{13} \right) \right) \right) \cosh^2(h) - \frac{3}{13} \left(\hat{P}_1^2 + \left| \hat{P}_1 \right|^2 + 2\hat{P}_1 \right) \left(-\text{COMB}_{3,1}^2 \left| \hat{P}_1 \right|^2 + \left(\left(\hat{P}_1 + 1 \right) \left| \text{COMB}_{3,1} \right|^2 \right. \right. \\
& \left. \left. - \text{COMB}_{3,1}^2 \right) \hat{P}_1 \right) \right) + \frac{19i \left(\hat{P}_1 + 1 \right) \hat{P}_1 \text{COMB}_{3,1}}{24} \left(\left(\frac{100\hat{P}_2 \left| \hat{P}_1 \right|^6}{19} \left(\frac{67}{100} + \hat{P}_1 - \frac{33\hat{P}_2}{100} \right) + \frac{146\hat{P}_1 \left| \hat{P}_1 \right|^4}{19} \left(-\frac{65 \left| \hat{P}_2 \right|^2}{146} \left(\frac{101}{130} + \hat{P}_1 - \frac{29\hat{P}_2}{130} \right) \right. \right. \right. \\
& \left. \left. \left. + \left(\frac{411}{292} + \hat{P}_1 \left(\hat{P}_1 - \frac{91\hat{P}_2}{146} \right) + \frac{218\hat{P}_1}{73} - \frac{351\hat{P}_2}{292} \right) \hat{P}_2 \right) + \frac{26\hat{P}_1^2 \left| \hat{P}_1 \right|^2}{19} \left(-\frac{34 \left| \hat{P}_2 \right|^2}{13} \left(\frac{147}{68} + \hat{P}_1 \left(\hat{P}_1 - \frac{11\hat{P}_2}{34} \right) + \frac{61\hat{P}_1}{17} - \frac{3}{4} \hat{P}_2 \right) \right. \right. \right. \\
& \left. \left. \left. + \hat{P}_2 \left(\frac{87}{13} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{8\hat{P}_2}{13} \right) + 11\hat{P}_1 \left(\hat{P}_1 - \frac{96\hat{P}_2}{143} \right) + \frac{252\hat{P}_1}{13} - \frac{123\hat{P}_2}{13} \right) \right) + \hat{P}_1^3 \left(-\frac{7 \left| \hat{P}_2 \right|^2}{19} \left(\frac{100}{7} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{\hat{P}_2}{2} \right) + \frac{171\hat{P}_1}{14} \left(\hat{P}_1 - \frac{58\hat{P}_2}{171} \right) \right. \right. \right. \\
& \left. \left. \left. + \frac{193\hat{P}_1}{7} - \frac{40\hat{P}_2}{7} \right) \left(\frac{32}{19} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{25\hat{P}_2}{38} \right) + \frac{245\hat{P}_1}{38} \left(\hat{P}_1 - \frac{188\hat{P}_2}{245} \right) + \frac{154\hat{P}_1}{19} - \frac{100\hat{P}_2}{19} \right) \hat{P}_2 \right) \right) \cosh^8(h) + \left(-\frac{260\hat{P}_2 \left| \hat{P}_1 \right|^6}{19} \left(\frac{97}{130} + \hat{P}_1 \right. \right. \\
& \left. \left. - \frac{33\hat{P}_2}{130} \right) - \frac{425\hat{P}_1 \left| \hat{P}_1 \right|^4}{19} \left(\left(-\frac{167\hat{P}_1}{850} + \frac{79\hat{P}_2}{1700} - \frac{3}{20} \right) \left| \hat{P}_2 \right|^2 + \left(\frac{3267}{1700} + \hat{P}_1 \left(\hat{P}_1 - \frac{253\hat{P}_2}{850} \right) + \frac{284\hat{P}_1}{85} - \frac{1219\hat{P}_2}{1700} \right) \hat{P}_2 \right) \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{59\hat{P}_1^2|\hat{P}_1|^2}{19}\left(-\frac{229|\hat{P}_2|^2}{118}\left(\frac{420}{229}+\hat{P}_1\left(\hat{P}_1-\frac{64\hat{P}_2}{229}\right)+\frac{728\hat{P}_1}{229}-\frac{143\hat{P}_2}{229}\right)+\left(\frac{2017}{118}+\hat{P}_1^2\left(\hat{P}_1-\frac{39\hat{P}_2}{118}\right)+\frac{893\hat{P}_1}{59}\left(\hat{P}_1-\frac{260\hat{P}_2}{893}\right)+\frac{2001\hat{P}_1}{59}\right.\right. \\
& \left.\left.-\frac{399\hat{P}_2}{59}\right)\hat{P}_2\right)-\frac{45\hat{P}_1^3}{19}\left(-\frac{14|\hat{P}_2|^2}{45}\left(\frac{78}{7}+\hat{P}_1^2\left(\hat{P}_1-\frac{17\hat{P}_2}{56}\right)+\frac{87\hat{P}_1}{8}\left(\hat{P}_1-\frac{54\hat{P}_2}{203}\right)+\frac{157\hat{P}_1}{7}-4\hat{P}_2\right)+\left(\frac{376}{45}+\hat{P}_1^2\left(\hat{P}_1-\frac{61\hat{P}_2}{180}\right)\right.\right. \\
& \left.\left.+\frac{1721\hat{P}_1}{180}\left(\hat{P}_1-\frac{500\hat{P}_2}{1721}\right)+\frac{323\hat{P}_1}{18}-\frac{52\hat{P}_2}{15}\right)\hat{P}_2\right)\right)\cosh^6(h)+\left(\frac{315\hat{P}_2|\hat{P}_1|^6}{38}\left(\frac{517}{630}+\hat{P}_1-\frac{113\hat{P}_2}{630}\right)+\frac{261\hat{P}_1|\hat{P}_1|^4}{19}\left(-\frac{25|\hat{P}_2|^2}{522}\left(\frac{43}{50}+\hat{P}_1\right.\right.\right. \\
& \left.\left.-\frac{7\hat{P}_2}{50}\right)+\left(\frac{206}{87}+\hat{P}_1\left(\hat{P}_1-\frac{20\hat{P}_2}{261}\right)+\frac{962\hat{P}_1}{261}-\frac{103\hat{P}_2}{261}\right)\hat{P}_2\right)+\frac{67\hat{P}_1^2|\hat{P}_1|^2}{38}\left(-\frac{127|\hat{P}_2|^2}{134}\left(\frac{223}{127}+\hat{P}_1\left(\hat{P}_1+\frac{10\hat{P}_2}{127}\right)+\frac{364\hat{P}_1}{127}-\frac{4\hat{P}_2}{127}\right)\right. \\
& \left.\left.+\hat{P}_2\left(\frac{3379}{134}+\hat{P}_1^2\left(\hat{P}_1-\frac{11\hat{P}_2}{134}\right)+\frac{1176\hat{P}_1}{67}\left(\hat{P}_1-\frac{4\hat{P}_2}{49}\right)+\frac{2961\hat{P}_1}{67}-\frac{253\hat{P}_2}{67}\right)\right)\right)+\frac{107\hat{P}_1^3}{76}\left(-\frac{27|\hat{P}_2|^2}{107}\left(\frac{68}{9}+\hat{P}_1^2\left(\hat{P}_1-\frac{\hat{P}_2}{9}\right)\right.\right. \\
& \left.\left.+\frac{205\hat{P}_1}{27}\left(\hat{P}_1+\frac{4\hat{P}_2}{205}\right)+\frac{389\hat{P}_1}{27}\right)+\hat{P}_2\left(\frac{1400}{107}+\hat{P}_1^2\left(\hat{P}_1-\frac{8\hat{P}_2}{107}\right)+\frac{1230\hat{P}_1}{107}\left(\hat{P}_1-\frac{53\hat{P}_2}{615}\right)+\frac{2629\hat{P}_1}{107}-\frac{204\hat{P}_2}{107}\right)\right)\right)\cosh^4(h) \\
& +\left(\frac{12\hat{P}_2|\hat{P}_1|^6}{19}\left(\frac{31}{48}+\hat{P}_1-\frac{17\hat{P}_2}{48}\right)+\frac{179\hat{P}_1|\hat{P}_1|^4}{76}\left(\left(-\frac{19\hat{P}_1}{179}-\frac{19}{179}\right)|\hat{P}_2|^2+\left(\frac{193}{179}+\hat{P}_1\left(\hat{P}_1-\frac{60\hat{P}_2}{179}\right)+\frac{423\hat{P}_1}{179}-\frac{111\hat{P}_2}{179}\right)\hat{P}_2\right)\right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{5\hat{P}_1^2|\hat{P}_1|^2}{76} \left(\left(-12\hat{P}_1^2 - \frac{158\hat{P}_1}{5} - \frac{98}{5} \right) |\hat{P}_2|^2 + \left(47 + \hat{P}_1^2 \left(\hat{P}_1 - \frac{3}{5}\hat{P}_2 \right) + 62\hat{P}_1 \left(\hat{P}_1 - \frac{63\hat{P}_2}{155} \right) + \frac{591\hat{P}_1}{5} - \frac{174\hat{P}_2}{5} \right) \hat{P}_2 \right) \\
& + \frac{1}{19} \hat{P}_1^3 \left(\left(-\frac{1}{4}\hat{P}_1^3 - \frac{63\hat{P}_1^2}{4} - \frac{71\hat{P}_1}{2} - 20 \right) |\hat{P}_2|^2 + \left(18 + \hat{P}_1^2 \left(\hat{P}_1 - \frac{3}{4}\hat{P}_2 \right) + 32\hat{P}_1 \left(\hat{P}_1 - \frac{33\hat{P}_2}{64} \right) + \frac{213\hat{P}_1}{4} - 20\hat{P}_2 \right) \hat{P}_2 \right) \cosh^2(h) \\
& - \frac{(7\hat{P}_1 + 7) \left(|\hat{P}_1|^2 + \hat{P}_1 \right) \hat{P}_2 \left(\frac{89|\hat{P}_1|^4}{14} + \left(\frac{233\hat{P}_1^2}{14} + \frac{411\hat{P}_1}{14} \right) |\hat{P}_1|^2 + \hat{P}_1^2 \left(\hat{P}_1^2 + \frac{261\hat{P}_1}{14} + 24 \right) \right)}{76} \Bigg) |\hat{P}_1 + 1| \\
& + \frac{1}{12} i \hat{P}_1 \sinh(h) \cosh(h) \left(\hat{P}_1 + 1 \right) \left(2 \cosh^2(h) \hat{P}_1 - \cosh^2(h) \hat{P}_2 + \cosh^2(h) - \hat{P}_1 - 1 \right) \left(|\hat{P}_1|^2 + \hat{P}_1 \right) \\
& \times \left(2 |\hat{P}_1|^2 \cosh^2(h) \hat{P}_2 - |\hat{P}_2|^2 \cosh^2(h) \hat{P}_1 - |\hat{P}_1|^2 \hat{P}_2 + \hat{P}_2 \hat{P}_1 \cosh^2(h) - \hat{P}_1 \hat{P}_2 \right) \left(-13 \text{COMB}_{3,1}^2 |\hat{P}_1|^2 \cosh^2(h) \right. \\
& - 3 \text{COMB}_{3,1}^2 \cosh^2(h) \hat{P}_1^2 + 3 |\hat{P}_1|^2 |\text{COMB}_{3,1}|^2 \cosh^2(h) + 13 |\text{COMB}_{3,1}|^2 \cosh^2(h) \hat{P}_1^2 + 10 \text{COMB}_{3,1}^2 |\hat{P}_1|^2 \\
& \left. - 16 \text{COMB}_{3,1}^2 \cosh^2(h) \hat{P}_1 + 16 |\text{COMB}_{3,1}|^2 \cosh^2(h) \hat{P}_1 - 10 |\text{COMB}_{3,1}|^2 \hat{P}_1^2 + 10 \text{COMB}_{3,1}^2 \hat{P}_1 - 10 |\text{COMB}_{3,1}|^2 \hat{P}_1 \right) \omega \\
& - \frac{67i(\hat{P}_1 + 1) \left(|\hat{P}_1|^2 + \hat{P}_1 \right) \text{COMB}_{3,1}}{24} \left(\left(\frac{26\hat{P}_2|\hat{P}_1|^6}{67} \left(\frac{19}{26} + \hat{P}_1 - \frac{7\hat{P}_2}{26} \right) + \frac{146\hat{P}_1|\hat{P}_1|^4}{67} \left(-\frac{8|\hat{P}_2|^2}{73} \left(\frac{25}{32} + \hat{P}_1 - \frac{7\hat{P}_2}{32} \right) \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{245}{292} + \hat{P}_1 \left(\hat{P}_1 - \frac{34\hat{P}_2}{73} \right) + \frac{143\hat{P}_1}{73} - \frac{171\hat{P}_2}{292} \right) \hat{P}_2 \Bigg) + \frac{100\hat{P}_1^2 |\hat{P}_1|^2}{67} \left(-\frac{91|\hat{P}_2|^2}{100} \left(\frac{94}{91} + \hat{P}_1 \left(\hat{P}_1 - \frac{22\hat{P}_2}{91} \right) + \frac{192\hat{P}_1}{91} - \frac{29\hat{P}_2}{91} \right) \right. \\
& + \left(\frac{77}{50} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{13\hat{P}_2}{20} \right) + \frac{109\hat{P}_1}{25} \left(\hat{P}_1 - \frac{61\hat{P}_2}{109} \right) + \frac{126\hat{P}_1}{25} - \frac{193\hat{P}_2}{100} \right) \hat{P}_2 \Bigg) + \hat{P}_1^3 \left(-\frac{33|\hat{P}_2|^2}{67} \left(\frac{100}{33} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{29\hat{P}_2}{66} \right) \right. \right. \\
& + \left. \left. \frac{117\hat{P}_1}{22} \left(\hat{P}_1 - \frac{34\hat{P}_2}{117} \right) + \frac{82\hat{P}_1}{11} - \frac{40\hat{P}_2}{33} \right) + \hat{P}_2 \left(\frac{32}{67} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{101\hat{P}_2}{134} \right) + \frac{411\hat{P}_1}{134} \left(\hat{P}_1 - \frac{98\hat{P}_2}{137} \right) + \frac{174\hat{P}_1}{67} - \frac{100\hat{P}_2}{67} \right) \right) \Bigg) \cosh^8(h) \\
& + \left(-\frac{59\hat{P}_2 |\hat{P}_1|^6}{67} \left(\frac{45}{59} + \hat{P}_1 - \frac{14\hat{P}_2}{59} \right) - \frac{425\hat{P}_1 |\hat{P}_1|^4}{67} \left(-\frac{39|\hat{P}_2|^2}{850} \left(\frac{61}{78} + \hat{P}_1 - \frac{17\hat{P}_2}{78} \right) + \left(\frac{1721}{1700} + \hat{P}_1 \left(\hat{P}_1 - \frac{229\hat{P}_2}{850} \right) + \frac{893\hat{P}_1}{425} \right. \right. \right. \\
& - \left. \left. \frac{609\hat{P}_2}{1700} \right) \hat{P}_2 \right) - \frac{260\hat{P}_1^2 |\hat{P}_1|^2}{67} \left(\left(-\frac{253\hat{P}_1}{520} \left(\hat{P}_1 - \frac{64\hat{P}_2}{253} \right) - \hat{P}_1 + \frac{81\hat{P}_2}{520} - \frac{25}{52} \right) |\hat{P}_2|^2 + \left(\frac{323}{104} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{167\hat{P}_2}{520} \right) + \frac{71\hat{P}_1}{13} \left(\hat{P}_1 - \frac{91\hat{P}_2}{355} \right) \right. \right. \\
& + \left. \left. \frac{2001\hat{P}_1}{260} - \frac{157\hat{P}_2}{130} \right) \hat{P}_2 \right) - \frac{194\hat{P}_1^3}{67} \left(-\frac{33|\hat{P}_2|^2}{97} \left(\frac{26}{11} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{79\hat{P}_2}{264} \right) + \frac{1219\hat{P}_1}{264} \left(\hat{P}_1 - \frac{286\hat{P}_2}{1219} \right) + \frac{133\hat{P}_1}{22} - \frac{28\hat{P}_2}{33} \right) + \left(\frac{188}{97} + \hat{P}_1^2 \left(\hat{P}_1 \right. \right. \right. \\
& - \left. \left. \frac{255\hat{P}_2}{776} \right) + \frac{3267\hat{P}_1}{776} \left(\hat{P}_1 - \frac{280\hat{P}_2}{1089} \right) + \frac{2017\hat{P}_1}{388} - \frac{78\hat{P}_2}{97} \right) \hat{P}_2 \Bigg) \cosh^6(h) + \left(\frac{1}{2} \left(\frac{107}{134} + \hat{P}_1 - \frac{27\hat{P}_2}{134} \right) \hat{P}_2 |\hat{P}_1|^6 + \frac{261\hat{P}_1 |\hat{P}_1|^4}{67} \left(-\frac{11|\hat{P}_2|^2}{1044} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{8}{11} + \hat{P}_1 - \frac{3}{11} \hat{P}_2 \right) + \left(\frac{205}{174} + \hat{P}_1 \left(\hat{P}_1 - \frac{127\hat{P}_2}{1044} \right) + \frac{196\hat{P}_1}{87} - \frac{205\hat{P}_2}{1044} \right) \hat{P}_2 \Bigg) + \frac{315\hat{P}_1^2 |\hat{P}_1|^2}{134} \left(-\frac{8|\hat{P}_2|^2}{63} \left(\frac{53}{40} + \hat{P}_1 \left(\hat{P}_1 + \frac{\hat{P}_2}{8} \right) + \frac{12\hat{P}_1}{5} \right. \right. \\
& + \left. \left. \frac{\hat{P}_2}{20} \right) + \hat{P}_2 \left(\frac{2629}{630} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{5\hat{P}_2}{63} \right) + \frac{1924\hat{P}_1}{315} \left(-\frac{7\hat{P}_2}{74} + \hat{P}_1 \right) + \frac{47\hat{P}_1}{5} - \frac{389\hat{P}_2}{630} \right) \right) + \frac{517\hat{P}_1^3}{268} \left(-\frac{113|\hat{P}_2|^2}{517} \left(\frac{204}{113} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{7\hat{P}_2}{113} \right) \right. \right. \\
& + \left. \left. \frac{412\hat{P}_1}{113} \left(\hat{P}_1 - \frac{\hat{P}_2}{103} \right) + \frac{506\hat{P}_1}{113} \right) + \left(\frac{1400}{517} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{43\hat{P}_2}{517} \right) + \frac{2472\hat{P}_1}{517} \left(\hat{P}_1 - \frac{223\hat{P}_2}{2472} \right) + \frac{3379\hat{P}_1}{517} - \frac{204\hat{P}_2}{517} \right) \hat{P}_2 \right) \Bigg) \cosh^4(h) \\
& + \left(\frac{(4 + 5\hat{P}_1 - \hat{P}_2)\hat{P}_2 |\hat{P}_1|^6}{268} + \frac{179\hat{P}_1 |\hat{P}_1|^4}{268} \left(\left(-\frac{3\hat{P}_1}{179} - \frac{3}{179} \right) |\hat{P}_2|^2 + \left(\frac{128}{179} + \hat{P}_1 \left(\hat{P}_1 - \frac{60\hat{P}_2}{179} \right) + \frac{310\hat{P}_1}{179} - \frac{63\hat{P}_2}{179} \right) \hat{P}_2 \right) \right. \\
& + \left. \frac{12\hat{P}_1^2 |\hat{P}_1|^2}{67} \left(\left(-\frac{5}{4}\hat{P}_1^2 - \frac{21\hat{P}_1}{8} - \frac{11}{8} \right) |\hat{P}_2|^2 + \hat{P}_2 \left(\frac{71}{16} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{19\hat{P}_2}{48} \right) + \frac{141\hat{P}_1}{16} \left(\hat{P}_1 - \frac{158\hat{P}_2}{423} \right) + \frac{197\hat{P}_1}{16} - \frac{71\hat{P}_2}{24} \right) \right) \right) \\
& + \left. \frac{31\hat{P}_1^3}{268} \left(\left(-\frac{17\hat{P}_1^3}{31} - \frac{111\hat{P}_1^2}{31} - \frac{174\hat{P}_1}{31} - \frac{80}{31} \right) |\hat{P}_2|^2 + \hat{P}_2 \left(\frac{72}{31} + \hat{P}_1^2 \left(\hat{P}_1 - \frac{19\hat{P}_2}{31} \right) + \frac{193\hat{P}_1}{31} \left(\hat{P}_1 - \frac{98\hat{P}_2}{193} \right) + \frac{235\hat{P}_1}{31} - \frac{80\hat{P}_2}{31} \right) \right) \right) \Bigg) \cosh^2(h)
\end{aligned}$$

$$\begin{aligned}
& - \frac{(89\hat{P}_1 + 89)\left(|\hat{P}_1|^2 + \hat{P}_1\right)\hat{P}_2\left(\frac{14|\hat{P}_1|^4}{89} + \left(\frac{233\hat{P}_1^2}{89} + \frac{261\hat{P}_1}{89}\right)|\hat{P}_1|^2 + \hat{P}_1^2\left(\hat{P}_1^2 + \frac{411\hat{P}_1}{89} + \frac{336}{89}\right)\right)}{536} \Bigg) \Bigg) \\
& \times \left((\hat{P}_1 + 1) \cosh(h) \left(\left(2|\hat{P}_1|^2 \hat{P}_2 + \hat{P}_1 \left(-|\hat{P}_2|^2 + \hat{P}_2 \right) \right) \cosh^2(h) - \hat{P}_2 \left(|\hat{P}_1|^2 + \hat{P}_1 \right) \right) \left(-\frac{1}{3} \hat{P}_1 \left(13|\hat{P}_1|^2 + 3\hat{P}_1^2 + 16\hat{P}_1 \right) |\hat{P}_1 + 1| \right. \right. \\
& \left. \left. + \frac{1}{3} \left(|\hat{P}_1|^2 + \hat{P}_1 \right) \left(3|\hat{P}_1|^2 + 13\hat{P}_1^2 + 16\hat{P}_1 \right) \right) \hat{P}_1 (\cosh(h) + 1) \left((1 + 2\hat{P}_1 - \hat{P}_2) \cosh^2(h) - \hat{P}_1 - 1 \right) \sinh(h) (\cosh(h) - 1) \right)^{-1} \quad (2.6)
\end{aligned}$$

3. The full $C_{4,2}$ expression

Here, we give the full expression for $C_{4,2}$, defined in (A 45) as the coefficient of the $\mathcal{O}(\varepsilon^4)$ correction to the $m = 2$ harmonic of η .

$C_{4,2} =$

$$-8 \cosh(h) \left(\left(-\frac{1}{72} \left(\hat{P}_1 \text{COMB}_{3,1} \sinh(h) \cosh(h) \left(12 \cosh^2(h) \hat{P}_1 - 4 \cosh^2(h) \hat{P}_3 + 8 \cosh^2(h) - 9 \hat{P}_1 + \hat{P}_3 - 8 \right) \left(6 \cosh^4(h) \hat{P}_1 \right. \right. \right.
\right.$$

$$\begin{aligned}
& + \cosh^4(h) \hat{P}_2 + 7 \cosh^4(h) + \cosh^2(h) \hat{P}_1 + \cosh^2(h) - 2\hat{P}_1 - 2 \Big) \omega \Big) - \frac{4}{3} \left(\frac{1}{8} \left(\frac{8}{9} + \hat{P}_1^2 + \left(\frac{\hat{P}_2}{12} - \frac{2}{9} \hat{P}_3 \right) \hat{P}_1 + \frac{1}{36} \hat{P}_2 \hat{P}_3 + \frac{67\hat{P}_1}{36} + \frac{\hat{P}_2}{9} \right. \right. \\
& - \frac{7\hat{P}_3}{36} \Big) \Big| \hat{P}_1 \Big| + \left(\frac{145}{144} + \left(\hat{P}_1^2 + \left(-\frac{5\hat{P}_3}{24} + \frac{\hat{P}_2}{16} \right) \hat{P}_1 + \frac{1}{24} \hat{P}_2 \hat{P}_3 \right) \hat{P}_1 + \frac{143\hat{P}_1^2}{48} + \left(\frac{17\hat{P}_2}{96} - \frac{29\hat{P}_3}{72} \right) \hat{P}_1 + \frac{13\hat{P}_2 \hat{P}_3}{288} + \frac{17\hat{P}_2}{144} - \frac{55\hat{P}_3}{288} \right. \\
& \left. \left. + \frac{859\hat{P}_1}{288} \right) \hat{P}_1 \right) \cosh^8(h) - \frac{4}{3} \left(-\frac{1}{6} \left(\frac{25}{32} + \hat{P}_1^2 + \left(-\frac{5\hat{P}_2}{64} - \frac{\hat{P}_3}{6} \right) \hat{P}_1 + \frac{5\hat{P}_2 \hat{P}_3}{192} + \frac{337\hat{P}_1}{192} - \frac{5\hat{P}_2}{96} - \frac{9\hat{P}_3}{64} \right) \Big| \hat{P}_1 \Big|^2 - \frac{53\hat{P}_1}{32} \left(\frac{37}{53} + \left(\hat{P}_1^2 + \right. \right. \right. \\
& \left. \left. \left(-\frac{34\hat{P}_3}{159} - \frac{49\hat{P}_2}{212} \right) \hat{P}_1 + \frac{41\hat{P}_2 \hat{P}_3}{636} \right) \hat{P}_1 + \frac{563\hat{P}_1^2}{212} + \left(-\frac{43\hat{P}_2}{106} - \frac{725\hat{P}_3}{1908} \right) \hat{P}_1 + \frac{32\hat{P}_2 \hat{P}_3}{477} + \frac{2243\hat{P}_1}{954} - \frac{82\hat{P}_2}{477} - \frac{26\hat{P}_3}{159} \right) \Big) \cosh^6(h) \\
& - \frac{4}{3} \left(\frac{13 \Big| \hat{P}_1 \Big|^2}{384} \left(\frac{8}{13} + \hat{P}_1^2 + \left(-\frac{6\hat{P}_2}{13} + \frac{2\hat{P}_3}{39} \right) \hat{P}_1 + \frac{1}{39} \hat{P}_2 \hat{P}_3 + \frac{62\hat{P}_1}{39} - \frac{17\hat{P}_2}{39} + \frac{\hat{P}_3}{13} \right) + \frac{59\hat{P}_1}{64} \left(\frac{30}{59} + \hat{P}_1 \left(\hat{P}_1^2 + \left(-\frac{21\hat{P}_3}{118} - \frac{45\hat{P}_2}{118} \right) \hat{P}_1 \right. \right. \right. \\
& \left. \left. \left. + \frac{8\hat{P}_2 \hat{P}_3}{177} \right) + \frac{877\hat{P}_1^2}{354} + \left(-\frac{130\hat{P}_2}{177} - \frac{164\hat{P}_3}{531} \right) \hat{P}_1 + \frac{49\hat{P}_2 \hat{P}_3}{1062} + \frac{1054\hat{P}_1}{531} - \frac{187\hat{P}_2}{531} - \frac{23\hat{P}_3}{177} \right) \Big) \cosh^4(h) + \frac{(19\hat{P}_1 + 19) \cosh^2(h)}{96} \left(-\frac{3 \Big| \hat{P}_1 \Big|^2}{19} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{25}{27} + \hat{P}_1 - \frac{2\hat{P}_3}{27} \right) + \left(\frac{113}{171} + \left(\hat{P}_1 - \frac{11\hat{P}_3}{57} \right) \hat{P}_1 + \frac{94\hat{P}_1}{57} - \frac{31\hat{P}_3}{171} \right) \hat{P}_1 \Bigg) + \frac{1}{16} (\hat{P}_1 + 1)^2 \left(\frac{1}{8} |\hat{P}_1|^2 + \left(\frac{35}{36} + \hat{P}_1 - \frac{11\hat{P}_3}{72} \right) \hat{P}_1 \right) \Bigg) |\hat{P}_1 + 1| \\
& + \frac{1}{288} \text{COMB}_{3,1} \sinh(h) \cosh(h) (\hat{P}_1 + 1) \left(12 \cosh^2(h) \hat{P}_1 - 4 \cosh^2(h) \hat{P}_3 + 8 \cosh^2(h) - 9\hat{P}_1 + \hat{P}_3 - 8 \right) \left(4 \cosh^4(h) |\hat{P}_1|^2 + 36 \cosh^4(h) \hat{P}_1^2 \right. \\
& + 12 \cosh^4(h) \hat{P}_1 \hat{P}_2 + 52 \cosh^4(h) \hat{P}_1 + 12 \cosh^2(h) \hat{P}_1^2 + 12 \cosh^2(h) \hat{P}_1 - |\hat{P}_1|^2 - 15\hat{P}_1^2 - 16\hat{P}_1 \Big) \omega - \frac{1}{12} (\hat{P}_1 + 1) \left(\left(-12 \left(\frac{8}{9} + \hat{P}_1^2 + \left(\frac{\hat{P}_2}{12} \right. \right. \right. \right. \\
& \left. \left. \left. - \frac{2}{9} \hat{P}_3 \right) \hat{P}_1 + \frac{1}{36} \hat{P}_2 \hat{P}_3 + \frac{67\hat{P}_1}{36} + \frac{\hat{P}_2}{9} - \frac{7\hat{P}_3}{36} \right) |\hat{P}_1| + \hat{P}_1 \left(-\frac{25}{3} + \left(\hat{P}_1 - \frac{\hat{P}_3}{3} \right) \left(\hat{P}_1 + \frac{5}{2} \hat{P}_2 \right) \hat{P}_1 - \frac{41\hat{P}_1^2}{6} + \left(\frac{19\hat{P}_2}{6} + \frac{7}{6} \hat{P}_3 \right) \hat{P}_1 - \frac{7}{6} \hat{P}_2 \hat{P}_3 + \frac{\hat{P}_2}{3} \right. \right. \\
& \left. \left. + \frac{7}{6} \hat{P}_3 - \frac{95\hat{P}_1}{6} \right) \right) \cosh^8(h) + \left(17 \left(\frac{151}{204} + \hat{P}_1^2 + \left(-\frac{7\hat{P}_2}{68} - \frac{19\hat{P}_3}{102} \right) \hat{P}_1 + \frac{1}{34} \hat{P}_2 \hat{P}_3 + \frac{349\hat{P}_1}{204} - \frac{5\hat{P}_2}{68} - \frac{8\hat{P}_3}{51} \right) |\hat{P}_1|^2 + \frac{17\hat{P}_1}{4} \left(\frac{106}{51} + \left(\hat{P}_1^2 + \right. \right. \right. \\
& \left. \left. \left(-\frac{19\hat{P}_3}{51} - \frac{36\hat{P}_2}{17} \right) \hat{P}_1 + \frac{31\hat{P}_2 \hat{P}_3}{51} \right) \hat{P}_1 + \frac{230\hat{P}_1^2}{51} + \left(-\frac{206\hat{P}_2}{51} - \frac{15\hat{P}_3}{17} \right) \hat{P}_1 + \frac{37\hat{P}_2 \hat{P}_3}{51} + \frac{93\hat{P}_1}{17} - \frac{92\hat{P}_2}{51} - \frac{20\hat{P}_3}{51} \right) \Bigg) \cosh^6(h) + \left(-\frac{19|\hat{P}_1|^2}{4} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{34}{57} + \hat{P}_1^2 + \left(-\frac{15\hat{P}_2}{38} - \frac{5\hat{P}_3}{114} \right) \hat{P}_1 + \frac{2\hat{P}_2\hat{P}_3}{57} + \frac{89\hat{P}_1}{57} - \frac{41\hat{P}_2}{114} - \frac{\hat{P}_3}{114} \right) - \left(\frac{131\hat{P}_1}{8} \left(\frac{200}{393} + \left(\hat{P}_1^2 + \left(-\frac{40\hat{P}_3}{131} - \frac{54\hat{P}_2}{131} \right) \hat{P}_1 + \frac{7\hat{P}_2\hat{P}_3}{131} \right) \hat{P}_1 \right. \right. \\
& \left. \left. + \frac{337\hat{P}_1^2}{131} + \left(-\frac{116\hat{P}_2}{131} - \frac{224\hat{P}_3}{393} \right) \hat{P}_1 + \frac{25\hat{P}_2\hat{P}_3}{393} + \frac{814\hat{P}_1}{393} - \frac{182\hat{P}_2}{393} - \frac{100\hat{P}_3}{393} \right) \right) \cosh^4(h) + \frac{(99\hat{P}_1 + 99) \cosh^2(h)}{8} \left(-\frac{29|\hat{P}_1|^2}{198} \left(\frac{80}{87} + \hat{P}_1 \right. \right. \\
& \left. \left. - \frac{7\hat{P}_3}{87} \right) + \hat{P}_1 \left(\frac{71}{99} + \hat{P}_1 \left(\hat{P}_1 - \frac{4\hat{P}_3}{27} \right) + \frac{1013\hat{P}_1}{594} - \frac{3\hat{P}_3}{22} \right) \right) - \frac{9\hat{P}_1 + 9}{8} \left(-\frac{7|\hat{P}_1|^2}{12} \left(\hat{P}_1 - \frac{4\hat{P}_3}{63} + \frac{59}{63} \right) + \left(\hat{P}_1 \left(\hat{P}_1 - \frac{\hat{P}_3}{36} \right) + \frac{25\hat{P}_1}{18} + \frac{\hat{P}_3}{108} \right. \right. \\
& \left. \left. + \frac{23}{54} \right) \hat{P}_1 \right) \right) \left(\left(\left(\frac{1}{18} \left(-54\hat{P}_1^2 + 4\hat{P}_1\hat{P}_2 - 2|\hat{P}_1|^2 - 52\hat{P}_1 \right) \cosh^2(h) + \frac{3}{2}\hat{P}_1^2 + \frac{1}{18}|\hat{P}_1|^2 + \frac{14\hat{P}_1}{9} \right) |\hat{P}_1 + 1| + \frac{1}{6}(\hat{P}_1 + 1) \left(6 \cosh^2(h) |\hat{P}_1|^2 \right. \right. \right. \\
& \left. \left. + 18 \cosh^2(h) \hat{P}_1^2 - 4\hat{P}_2\hat{P}_1 \cosh^2(h) + 20 \cosh^2(h) \hat{P}_1 - 3|\hat{P}_1|^2 - 9\hat{P}_1^2 - 12\hat{P}_1 \right) \right)^{-1} \left(\left((1 - \hat{P}_2 + 2\hat{P}_1) \cosh^2(h) - \hat{P}_1 - 1 \right) \sinh(h) \right. \\
& \left. \times (\cosh(h) + 1)(\cosh(h) - 1) \left((8 + 12\hat{P}_1 - 4\hat{P}_3) \cosh^2(h) - 8 - 9\hat{P}_1 + \hat{P}_3 \right) \right)^{-1}
\end{aligned} \tag{3.1}$$