31 Diagram Chases
for your amusement

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How to play

In every chase, you will be presented with an image called a **commutative diagram**, looking something like this:

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, "j"] & D
\end{tikzcd}
$$

In the mathematical fields where diagram chasing arises, category theory and homological algebra, the capital letters represent various abstract structures, but for our purposes, they can be thought of as sets of numbers. The arrows represent functions that take as input something from the set they start at and output something from the set they point towards. For example, in the above diagram, f would be a function with inputs in A and outputs in B. So if k is some number in A, then $f(k)$ is some number in B.

The diagram is called "commutative" because following any "path" around the diagram will provide the same result regardless of path chosen, so long as the start and end points are the same. For example, in the diagram above, following the f arrow then the h arrow will take something from A to something in D. Following

g then j also goes from A to D, so commutativity guarantees that applying f then h will result in the same element of D as applying g then j. In symbols, for any a in A,

$$h(f(a)) = j(g(a))$$

The functions we will be working with aren't any old functions, we'll be assuming all functions are homomorphisms, which in this context means that for any two inputs a and b,

$$f(a + b) = f(a) + f(b)$$

In particular, this means that $f(0) = 0$ for all our functions[1].

The goal of diagram chasing is to prove that certain functions have certain properties given that other functions have other properties. Most often, these properties will be **injectivity** and **surjectivity**.

A function is said to be injective if different inputs go to different outputs. Another way of saying this is that if $f(a) = f(b)$, then $a = b$. The fact that our functions are homomorphisms means that injectivity is equivalent to the property that the only input that gets sent to 0 is 0 itself.

A function is surjective if every number is an actual output of the function. So if some surjective function f goes from A to B, then for every b in B, there is some a in A such that $f(a) = b$.

A function is bijective if it is both injective and surjective.

Advanced properties

The **image** of a homomorphism $A \xrightarrow{f} B$, written $\operatorname{Im} f$, is everything in B that f actually maps to. That is, the image is all b in B such that there is some a in A with $f(a) = b$. We can see immediately that f is surjective if and only if $\operatorname{Im} f = B$.

The **kernel** of f, written $\ker f$, is everything in A that gets sent to 0 by f. That is, it is the set of all a in A with $f(a) = 0$. This may seem like an odd thing to define, but notice that if x and y are in $\ker f$, then

$$f(x + y) = f(x) + f(y) = 0 + 0 = 0$$

so $x + y$ is in $\ker f$.

Finally, a pair of functions $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are said to be **exact** at B if $\operatorname{Im} f = \ker g$. A consequence of this is that $g(f(a)) = 0$ for all a in A.

[1] Proof: $f(0) + f(0) = f(0 + 0) = f(0)$, so subtracting $f(0)$ from both sides we get $f(0) = 0$.

Examples

So you can get an idea of how this works, we'll do two easy examples with the following diagram:

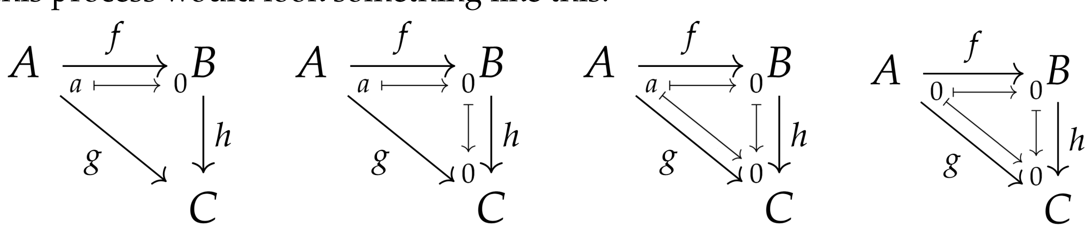
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$$

Ex. 1

If g is injective, prove f is injective[2].

To prove injectivity, we will assume a in A is sent to 0 in B by f, then prove a must itself be 0. By commutativity, we have $g(a) = h(f(a))$. But $f(a) = 0$, so $h(f(a)) = h(0) = 0$, since all homomorphisms send 0 to 0. Thus $g(a) = 0$, and since g is injective, this means $a = 0$, which is what we were trying to prove. Thus f is injective.

Often chases like this are not done (at least initially) in words or symbols, as we've done, but by writing elements of the sets on the diagram and following them around, tracking what maps to what, hence the name "chase". For this example, this process would look something like this:



It's highly recommended that you work through the chases like this rather than via a formal proof (although you're welcome to then translate the picture to words), so large images of the diagrams are provided for every chase, even when the diagram is the same as a previous one.

Ex. 2

If g is surjective and h is injective, prove f is surjective.

[2] This is actually true for all functions, not just homomorphisms, so the technique used here is slightly overkill, but it's more characteristic of the typical way of proving injectivity.

Try to work it out on the diagram yourself, if you get stuck some back here for a hint on what to do next.

To prove surjectivity, we let b be some arbitrary number in B and find something in A that f sends to b. First, we observe that $h(b)$ is some number in C. Since g is surjective, there is some a in A such that $g(a) = h(b)$. Then, by commutativity, $g(a) = h(f(a))$. But $g(a) = h(b)$, so $h(b) = h(f(a))$. Finally, injectivity of h tells us that $b = f(a)$. This tells us that everything in B is the output from f of something in A, meaning f is surjective.

Chases

Chase 1

$$\begin{tikzcd} A \arrow[r, two heads, "f"] \arrow[dr, hook, "g"'] & B \arrow[d, "h"] \\ & C \end{tikzcd}$$

If f is surjective and g is injective, prove h is injective.

Chase 2

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, two heads, "g"'] & B \arrow[d, "h"] \\
& C
\end{tikzcd}
$$

If g is surjective, prove h is surjective.

Chase 3

$$
\begin{tikzcd}
A \arrow[r, two heads, "f"] \arrow[dr, "g"'] & B \arrow[d, two heads, "h"] \\
& C
\end{tikzcd}
$$

If f and h are surjective, prove g is surjective.

Chase 4

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[dr, "g"'] & B \arrow[d, hook, "h"] \\ & C \end{tikzcd}$$

If f and h are injective, prove g is injective.

Remark. *Congratulations! You have just proven two useful mathematical results: compositions of injective functions are injective, and likewise for surjective functions. We'll now introduce another wrinkle: when we write* 0 *next to an arrow, we mean it is the zero function, the function that sends every input to 0.*

Chase 5

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "0"] \\ C \arrow[r, hook, "h"] & D \end{tikzcd}$$

If h is injective, prove g is the zero function.

Chase 6

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "g"'] & B \arrow[d, "0"] \\
C \arrow[r, "h"] & D
\end{tikzcd}
$$

If g is surjective, prove h is the zero function.

Chase 7

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[dl, "0"'] \\
C \arrow[r, "j"'] & D
\end{tikzcd}
$$

Prove both g and h are the zero function.

Remark. *There aren't many more interesting chases we can do with just these properties, we could come up with some convoluted diagram, but either it would be mostly superfluous or it would reduce to one of the above chases, or something close to them. So, it's time to introduce a key idea from homological algebra: exactness.*

Chase 8

$$A \xrightarrow{f} B \xrightarrow{g} C$$

If the row is exact (i.e. $\operatorname{Im} f = \ker g$), prove that f surjective implies g is the zero map and g injective implies f is the zero map.

Chase 9

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[dr, "h"] & \\
B \arrow[r, "i"'] & C \arrow[r, "j"'] & D
\end{tikzcd}$$

If the row along the bottom is exact (i.e. $\operatorname{Im} i = \ker j$), prove h is the zero function.

Chase 10

$$\begin{tikzcd}
& A \arrow[dr, "h"] & \\
B \arrow[ur, hook, "f"] \arrow[r, "i"'] & C \arrow[u, "g"] \arrow[r, "j"'] & D
\end{tikzcd}$$

If the row along the bottom is exact, and f is injective, prove g is injective.

Chase 11

$$
\begin{tikzcd}
 & A \arrow[d, "f"'] \arrow[dr, "0"] & \\
B \arrow[r, "g"] \arrow[dr, "0"'] & C \arrow[r, "h"'] \arrow[d, "i"] & D \\
 & E &
\end{tikzcd}
$$

If the row through the middle is exact, prove $i \circ f$, the function composed of first applying f, then i, is the zero map.

Chase 12

$$
\begin{tikzcd}
A \arrow[rr, "g"] & & B \arrow[d, hook, "h"] \\
C \arrow[u, two heads, "f"] \arrow[r, "i"'] & D \arrow[r, "j"'] & E
\end{tikzcd}
$$

If the row along the bottom is exact, f is surjective, and h is injective, prove g is the zero function.

Chase 13

$$
\begin{tikzcd}
 & A \arrow[dl, two heads, "f"'] \arrow[d, "g"] \arrow[dr, "h"] & \\
B \arrow[r, "i"'] & C \arrow[r, "j"'] & D
\end{tikzcd}
$$

If the row is exact, $\ker g = \ker h$, and f is surjective, prove i is the zero map.

Remark. *We're now entering into the realm of exercises performed in proper homological algebra. The following four chases appear in Dummit and Foote's Abstract Algebra, Section 10.5.*

Chase 14

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "h"'] & B \arrow[r, two heads, "g"] \arrow[d, hook, "i"] & C \arrow[d, "j"] \\
D \arrow[r, "k"'] & E \arrow[r, "l"'] & F
\end{tikzcd}
$$

If both rows are exact, g, h are surjective, and i is injective, prove j is injective.

Chase 15

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "h"'] & B \arrow[r, "g"] \arrow[d, "i"] & C \arrow[d, hook, "j"] \\
D \arrow[r, hook, "k"'] & E \arrow[r, "l"'] & F
\end{tikzcd}$$

If both rows are exact, and h, j, and k are injective, prove i is injective.

Chase 16

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "h"'] & B \arrow[r, two heads, "g"] \arrow[d, "i"] & C \arrow[d, two heads, "j"] \\
D \arrow[r, "k"'] & E \arrow[r, "l"'] & F
\end{tikzcd}$$

If both rows are exact, and g, h, and j are surjective, prove i is surjective.

Chase 17

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "g"] \arrow[d, two heads, "i"] & C \arrow[d, hook, "j"] \\ D \arrow[r, hook, "k"'] & E \arrow[r, "l"'] & F \end{tikzcd}$$

If both rows are exact, i is surjective, and j, k are injective, prove h is surjective.

Remark. *When we write* 0 *in place of a set* (*i.e. instead of a capital letter*), *we mean the set containing only* 0. *Any function into such a set is necessarily the zero function, as is any map out of it* (*since zero is always sent to zero*). *As such, we will usually leave these unlabeled. The diagram in the following puzzle is referred to as a* ***short exact sequence****, and is an important object of study in homological algebra.*

Chase 18

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

If the row is exact, prove that f is injective and g is surjective.

Chase 19 (The Strong Four Lemma)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "i"'] & B \arrow[r, "g"] \arrow[d, "j"'] & C \arrow[r, "h"] \arrow[d, "k"'] & D \arrow[d, hook, "l"'] \\
E \arrow[r, "m"'] & F \arrow[r, "n"'] & G \arrow[r, "p"'] & H
\end{tikzcd}$$

If the rows are exact, i is surjective, and l is injective, prove that $\ker k = g(\ker j)$ and $\operatorname{Im} j = n^{-1}(\operatorname{Im} k)$. Here, when we write a “a function of a set”, we mean the set made up of applying that function to everything in the set. So $g(\ker j)$ is the collection of outputs of g when you input everything that j maps to 0. $n^{-1}(\operatorname{Im} k)$ is the preimage of $\operatorname{Im} k$, meaning it is everything that n maps to something in $\operatorname{Im} k$.

Chase 20

$$\begin{tikzcd}
& A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[dd, "h"] & \\
B & & C \\
& D \arrow[ul, "i"] \arrow[ur, "j"'] &
\end{tikzcd}$$

If either f or g is surjective and either i or j is injective, prove that h is surjective.

Chase 21

$$
\begin{tikzcd}
 & A \arrow[d, "g"] & \\
C \arrow[ur, two heads, "f"] \arrow[r, "j"'] & D \arrow[r, "k"] \arrow[d, "h"'] & E \\
 & B \arrow[ur, hook, "i"'] &
\end{tikzcd}
$$

If the horizontal row is exact, f is surjective, and i is injective, prove that the column is exact.

Chase 22

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "g"] \arrow[dl, hook, "j"] & C \\
D \arrow[d, "i"'] & & \\
E \arrow[uurr, hook, "k"] & &
\end{tikzcd}$$

If the column on the left is exact and j, k are injective, prove that the row along the top is exact.

Bonus, if the directions of j and k are reversed, what properties do they need to have for the same conclusion to be true?

Chase 23 (The Short 5-Lemma)

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, hook, two heads, "h"'] & B \arrow[r, "g"] \arrow[d, "i"'] & C \arrow[r] \arrow[d, hook, two heads, "j"'] & 0 \\
0 \arrow[r] & D \arrow[r, "k"'] & E \arrow[r, "l"'] & F \arrow[r] & 0
\end{tikzcd}$$

If h and j are bijective and the rows are exact, prove that i is bijective.

Chase 24

$$
\begin{tikzcd}
A \arrow[dr, hook, "f"] & & C \\
& B \arrow[ur, "g"] \arrow[dr, "j"'] & \\
D \arrow[uu, "h"] \arrow[ur, "i"'] & & E \arrow[uu, "k"']
\end{tikzcd}
$$

If both diagonals are exact and f is injective, prove that h is surjective.

Chase 25

$$
\begin{tikzcd}
A \arrow[dr, "f"] & & C \\
& B \arrow[ur, "g"] \arrow[dr, two heads, "j"'] & \\
D \arrow[uu, "h"] \arrow[ur, "i"'] & & E \arrow[uu, "k"']
\end{tikzcd}
$$

If both diagonals are exact and j is surjective, prove that k is injective.

Chase 26

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "i"'] & C \arrow[d, hook, "j"'] \\
D \arrow[u, two heads, "h"] \arrow[r, "k"'] & E \arrow[r, "l"] \arrow[d, "m"'] & F \arrow[dl, "n"] \\
 & G &
\end{tikzcd}$$

If the top row and column in the middle are exact, j is injective, and h is surjective, prove the middle row is exact.

Chase 27

$$\begin{tikzcd}
A \arrow[r, "h"'] \arrow[rr, bend left, "f"] & B \arrow[r, "i"'] & C \arrow[r, "j"] \arrow[rr, bend right, "g"'] & D \arrow[r, "k"] & E
\end{tikzcd}$$

If the row as well as the pair of curved maps are exact, prove h is surjective and k is injective.

Chase 28 (The 5-Lemma)

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads, "l"] & B \arrow[r, "g"] \arrow[d, hook, two heads, "m"] & C \arrow[r, "h"] \arrow[d, "n"] & D \arrow[r, "j"] \arrow[d, hook, two heads, "p"] & E \arrow[d, hook, "q"] \\
F \arrow[r, "r"'] & G \arrow[r, "s"'] & H \arrow[r, "t"'] & I \arrow[r, "u"'] & J
\end{tikzcd}
$$

If the rows are exact, m and p are bijective, l is surjective, and q is injective, prove n is bijective.

Chase 29

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, two heads, "h"'] & & B \arrow[rr, "g"] \arrow[d, "i"'] & & C \arrow[dl, hook, "j"] \\
& D \arrow[r, "k"] \arrow[dr, "m"'] & E \arrow[r, "l"] \arrow[d, "n"] & F \arrow[dl, "o"] & \\
& & G & &
\end{tikzcd}
$$

If the top row, vertical column, and both diagonals are exact, h is surjective and j is injective, prove that the middle row is exact.

Chase 30 (The Nine (or 3×3) Lemma)

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "g"] \arrow[d, "i"'] & C \arrow[r] \arrow[d, "j"'] & 0 \\
0 \arrow[r] & E \arrow[r, "k"'] \arrow[d, "m"'] & F \arrow[r, "l"'] \arrow[d, "n"'] & G \arrow[r] \arrow[d, "p"'] & 0 \\
0 \arrow[r] & H \arrow[r, "q"'] \arrow[d] & I \arrow[r, "r"'] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

If the three columns and bottom two rows are exact, prove the top row is exact.

Chase 31 (**The Middle 3×3 Lemma**)

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r, "g"] \arrow[d, "i"'] & C \arrow[r] \arrow[d, "j"'] & 0 \\
0 \arrow[r] & E \arrow[r, "k"'] \arrow[d, "m"'] & F \arrow[r, "l"'] \arrow[d, "n"'] & G \arrow[r] \arrow[d, "p"'] & 0 \\
0 \arrow[r] & H \arrow[r, "q"'] \arrow[d] & I \arrow[r, "r"'] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the three columns, top row, and bottom row are exact, and $l \circ k = 0$, prove the middle row is exact.