Finite Spaces

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General

Proposition 1. *In a finite space X, every point is contained in a unique smallest neighborhood.*

Proof. The collection of neighborhoods around any point p is finite and nonempty (since it contains X), so there is some neighborhood with the minimal size. To see it is unique, just note that the intersection of any two distinct minimal neighborhoods of p is also a neighborhood of p and necessarily smaller than the minimal ones, contradicting minimality. $\square$

Definition 2. *We refer to this unique smallest neighborhood around p as the **minimal neighborhood of $\mathbf{p}$**.*

Proposition 3. *Let X be a finite space, and let $U_p \subset X$ be the minimal neighborhood of some $p \in X$.*

1. *U_p is connected.*
2. *Any open set containing p contains U_p.*
3. *X is Hausdorff if and only if it has the discrete topology, which is true if and only if $U_p = \{p\}$ for every $p \in X$.*
4. *A sequence (x_i) in X converges to p if and only if $x_i \in U_p$ for all sufficiently large i.*

Proof. Any disconnection of U_p would have to contain p in one of the two pieces, which would be a smaller neighborhood than U_p. If V is an open set containing p, then $V \cap U_p$ is strictly smaller than U_p unless $U_p \subset V$. Hausdorff iff no element is in another's minimal neighborhood iff $U_p = \{p\}$ for all p iff discrete. The last statement is also clear. $\square$

Proposition 4. *The collection of minimal neighborhoods of the points in a space is a basis for that space. In fact, it is the unique smallest basis of the space (i.e. contains the fewest basis sets).*

Proof. To be more explicit, let

$$\mathcal{B} = \{U_p \subset X \mid U_p \text{ is the minimal neighborhood of } p \in X\}.$$

To see $\mathcal{B}$ is a basis, we need only check that every open set $U \subset X$ is a union of sets in $\mathcal{B}$. And indeed, by the previous proposition, we have that the union of all the minimal neighborhoods of the points in U is exactly U.

Minimality follows from the evident fact that any basis of X must contain every minimal neighborhood. $\square$

Definition 5. *We call the basis made up of minimal neighborhoods the **minimal basis**.*

Proposition 6. *A map $f : X \to Y$ between finite topological spaces is continuous if and only if the minimal neighborhood of p is mapped into the minimal neighborhood of $f(p)$ for every $p \in X$.*

Proof. f is continuous if and only if the preimage of every basis set is open for any basis. In particular, choosing the minimal basis, we have continuity iff the preimage of every minimal neighborhood is open. This is the case iff every point in the preimage has its minimal neighborhood also contained in the preimage. In particular, the preimage of the minimal neighborhood of $f(p)$ must contain the minimal neighborhood of p. On the other hand, if the minimal neighborhood of p is always mapped into the minimal neighborhood of $f(p)$, then for any minimal B and any $p \in f^{-1}(B)$, we know the minimal neighborhood of $f(p)$ is a subset of B, so indeed the minimal neighborhood of p is in $f^{-1}(B)$. □

Connectedness and Paths

Lemma 7. *In any connected space X, there is at most one point $p \in X$ such that $X \setminus \{p\}$ is disconnected. In particular, it is always possible to remove a point and keep the space connected.*

Proof. Assume there are at least two such points, say p and q. Then let (A, B) and (C, D) be disconnections of $X \setminus \{p\}$ and $X \setminus \{q\}$ respectively. So

$$A \cup B \cup \{p\} = X = C \cup D \cup \{q\}.$$

In particular, we may assume without loss of generality that $q \in A$ and $p \in C$. We now claim that $A \cup C$ and $B \cap D$ disconnect X. Indeed, A and p are in the former, and B is split between C and D. So every point of X is either in $A \cup C$ or $B \cap D$. These are both open sets and are disjoint since $A \cap B = \varnothing$ and $C \cap D = \varnothing$, meaning nothing in both B and D is in either A or C. Hence we have a disconnection of X, which we assumed to be connected. □

Lemma 8. *Given any connected finite space X, any basis $\mathcal{B}$, and any $a, b \in X$, there exist $U_1, \ldots, U_n \in \mathcal{B}$ such that $a \in U_1$, $b \in U_n$ and $U_i \cap U_{i+1} \neq \varnothing$ for all i.*

Proof. By induction on the size of the space. If $\#X = 1$, the only possible basis is $\{X\}$, so just take $n = 1$ and $U_1 = X$. Now assume the conclusion for all spaces of size k, let $\#X = k + 1$, and fix a basis $\mathcal{B}$ for X.

By the previous lemma, there is some point $r \in X$ such that $X \setminus \{r\}$ is connected. Now, we may apply the inductive hypothesis to $X \setminus \{r\}$ with the basis $\mathcal{B}'$, the basis formed by removing r from any set in $\mathcal{B}$ containing it. So for any $p \in X \setminus \{r\}$ there is some collection of sets $U_i' \in \mathcal{B}'$ with $a \in U_1'$, $p \in U_n'$, and consecutive sets overlapping. Let U_i be the set in $\mathcal{B}$ from which r may or may not have been removed to get U_i'. For any $b = p \neq r$, we can just take these U_i's. But the interesting case is of course $b = r$.

There must be some basis set V containing r, assume first that it contains some other $p \in X$. Then take the U_i's as above and set $U_{n+1} = V$, and we're done.

Now we consider the case where the only set in $\mathcal{B}$ containing r is $\{r\}$. By connectedness, we must have that there is some $c \in X$ that is in no open set in X other than X itself. Otherwise, we could take the union of basis sets around each point other than r and disconnect X with that union and $\{r\}$. But $\mathcal{B}$ has to cover X, so we must have that $X \in \mathcal{B}$. So in this case we can just take $n = 1$ and $U_1 = X$. This covers every case so we're done. □

Theorem 9. *Every connected finite space is path-connected.*

Proof. Let $\mathcal{B}$ be the minimal basis for some connected finite space. Fix $a, b \in X$. By the lemma, there are $U_i \in \mathcal{B}$ with $a \in U_1$, $b \in U_n$, and $U_i \cap U_{i+1} \neq \varnothing$. For each U_i, let $x_i \in U_i$ be an element for which U_i is the minimal neighborhood. Also fix any $y_i \in U_i \cap U_{i+1}$ for each $1 \leq i < n$.

Now, define a map $\gamma : [0,1] \to X$ by

$$\gamma(t) = \begin{cases} a & 0 \le t < \frac{1}{n+1} \\ x_i & t = \frac{i}{n+1} \\ y_i & \frac{i}{n+1} < t < \frac{i+1}{n+1} \\ b & \frac{n}{n+1} < t \le 1 \end{cases}.$$

To see γ is continuous, we need only check the preimages of basis sets. As a result of how we chose the x_i's, we know x_i is in some $V \in \mathcal{B}$ if and only if $U_i \subset V$, in which case y_{i-1} and y_i are also in V. So, of the points we care about, V will contain at most x_i, y_i, and y_{i-1} for some values of i, and potentially also a and b. Thus $\gamma^{-1}(V)$ will be a union of open sets, namely

$$\gamma^{-1}(y_{i-1}) \cup \gamma^{-1}(x_i) \cup \gamma^{-1}(y_i) = \left(\frac{i-1}{n+1}, \frac{i+1}{n+1}\right),$$

for some i's, as well as $\gamma^{-1}(a)$, and $\gamma^{-1}(b)$. All of these are open, so γ is continuous, hence there is a path between a and b. □

Fundamental Group

Proposition 10. *Minimal neighborhoods are simply connected.*

Proof. Let $U \subset X$ be the minimal neighborhood of p, which we know is connected, hence path-connected. Let $a \in U$ be a point contained in its minimal neighborhood V inside U. If no such a exists, then U has the trivial topology, and any function into it is continuous, so every loop is null-homotopic. Let $\gamma : [0,1] \to X$ be any loop, which by path-connectedness we may assume is based at a.

So, there is some $\varepsilon > 0$ such that $\gamma([0,\varepsilon))$ and $\gamma((1-\varepsilon, 1])$ are both contained in V. We first define the following map from I to U:

$$\beta(s) = \begin{cases} a & 0 \le s < \varepsilon \\ p & \varepsilon \le s \le 1-\varepsilon \\ a & 1-\varepsilon < s \le 1 \end{cases}$$

Now, define $H : I \times I \to U$ by

$$H(s,t) = \begin{cases} \gamma(s) & t < \frac{1}{2} \\ \beta(s) & t = \frac{1}{2} \\ a & t > \frac{1}{2} \end{cases}$$

This is continuous since the "outer shell" composed of a and potentially other elements in V is open, as is the preimage of any open set that was mapped into by γ. We're free to use p to fill in gaps in the map because the only relevant open set p is in is U itself, the preimage of which will always be all of $I \times I$. So H is continuous, hence a path homotopy between γ and the constant path at a. So U is simply connected. □

Corollary 11. *Every finite space is locally simply connected, hence has a universal covering space if connected.*

Proof. Just take the minimal basis, which has only simply connected sets. □

Homotopic maps between finite spaces

There is a finite space $\mathbb{S}^n$ with $2n+2$ points that is weak homotopy equivalent to the usual n-sphere S^n (see May, Section 3.4 and Chapter 5). That is, the group of homotopy classes of maps from S^k to $\mathbb{S}^n$ is isomorphic to the group of homotopy classes of maps from S^k to $\mathbb{S}^n$. This naturally raises the question: what about maps from $\mathbb{S}^k$ to S^n and from $\mathbb{S}^k$ to $\mathbb{S}^n$? The former is boring, maps from connected finite spaces to Hausdorff spaces must be constant. The latter however, need not be trivial. In particular, for $k = n = 1$, we have that there are 5 homotopy classes (only three if you fix a basepoint).

Definition 12. *The **finite** n**-sphere** $\mathbb{S}^n$ is a set of $2n+2$ elements which we will denote $\{e_{i,\pm 1} \mid 1 \leq i \leq n+1\}$. The topology is the one induced by the preorder $e_{i,\pm 1} < e_{j,\pm 1}$ iff $i < j$. (In particular, $e_{i,+1}$ and $e_{i,-1}$ are incomparable). This is the space described in May, the reason for the notation we're using will be made clear in a proof below.*

There is a convenient characterization of homotopies between maps between finite spaces, so that they can be generated fairly easily (and in a finite amount of time). Someday I might write it down.

Finite Topological Groups

Certainly if we give any finite group the trivial or discrete topology it becomes a topological group (the multiplication and inversion maps will be out of and into discrete or trivial spaces, hence continuous).

Lemma 13. *The minimal neighborhoods in a finite topological group are all homeomorphic.*

Proof. If G is a finite topological group and $m : G \times G \to G$ is the multiplication function, then for any $a, b \in G$, the map $m(ba^{-1}, -) : G \to G$ is a homeomorphism (the inverse is $m(ab^{-1}, -)$). It must map the minimal neighborhood of a to some neighborhood of b, but if that neighborhood is strictly larger than the minimal neighborhood, then the preimage of the minimal neighborhood of b will be a smaller neighborhood around a. This can't be, so indeed the restriction of $m(ba^{-1}, -)$ to the minimal neighborhood of a is a homeomorphism onto the minimal neighborhood of b. □

Lemma 14. *All the minimal neighborhoods of a finite topological group are either disjoint or identical.*

Proof. They're all homeomorphic, so in particular they all have the same number of elements. Thus no minimal neighborhood can strictly contain another, nor can they have nonempty intersection. □

Proposition 15. *The minimal basis of a finite topological group is a partition into subsets which each contain the same number of points.*

Proof. From the lemmas. □

Corollary 16. *Every finite T_0 topological group (no indistinguishable points) has the discrete topology.*

Corollary 17. *Every finite topological group of prime order has either the trivial or discrete topology.*

Corollary 18. *Every finite topological space is either disconnected or has the trivial topology (but not both).*

Proposition 19. *If G is a finite group and $H \triangleleft G$ is a normal subgroup, then the topology generated by the cosets of H makes G a topological group.*

Proof. If m is the multiplication map, then $m^{-1}(gH)$ is just the union of all $aH \times bH \subset G \times G$ with $ab \in gH$. Each of these is open, so m is continuous. The inversion map i will map $gH \mapsto Hg^{-1} = g^{-1}H$ since H is normal. So the preimage of any basis set $i^{-1}(gH) = g^{-1}H$ is open. $\square$

Proposition 20. *If G is a finite topological group, then the minimal neighborhood of $1 \in G$ is a normal subgroup.*

Proof. Let A be the minimal neighborhood of 1. A is closed under inverses since i will restrict to a homeomorphism from A to A (since $1^{-1} = 1$). Similarly, for any $a \in A$, the map $m(a, -)$ will restrict to a homeomorphism from A to itself, since A is the minimal neighborhood of both 1 and a. For normality, just notice that the minimal neighborhood of ga for any $g \in G$ must contain g (since $m(g, -)$ is a homeomorphism from the minimal neighborhood of a to that of ga, which maps 1 to g). Then, $m(-, g^{-1})$ will be a homeomorphism from the minimal neighborhood of ga to that of gag^{-1}, but it maps g to 1 so $gag^{-1} \in A$. $\square$

Theorem 21. *There is a one-to-one correspondence between normal subgroups of a finite group and topologies on that group which make it a topological group. Also a homomorphism between finite topological groups is continuous if and only if it is well-defined as a map on the quotient.*

References

[1] J. P. May, *Finite spaces and larger contexts*, `https://math.uchicago.edu/~may/REU2024/FiniteSpaces24.pdf`