From self-similar structures to self-similar groups

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Outline

1. Background Information
   - Self-Similar Groups/Actions
   - Self-Similar Structures and P.C.F. Structures
2. Our Results
   - Limit Spaces of Self-Similar Groups
   - P.C.F. Structures on Limit Spaces
   - The Inverse Problem

Rooted Trees and Self-Similar Actions (Nekrashevych)

- A finite set X, called the *alphabet*
- Rooted tree structure of X^*, the set of all words
- An *automorphism* of X^* preserves adjacency of vertices
- A *self-similar group* G is a subgroup of Aut X^* that acts on X^* "letter by letter"
- G *contracting* if can be represented by a finite *Moore diagram*

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Moore Diagrams and Limit Spaces (Nekrashevych)

- Example: Binary adding machine

$$\mathsf{X} = \{0, 1\}, \ a(0) = 1, \ a(1) = 0, \ G = \langle a \rangle$$

Figure: Binary adding machine

$$\begin{aligned} a(11001) &= 0\ a(1001) \\ &= 00\ a(001) \\ &= 001\ e(01) \\ &= 00101 \end{aligned}$$

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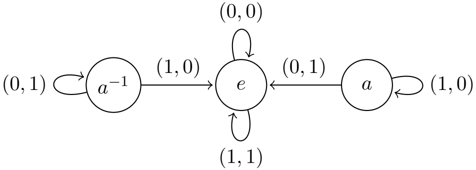

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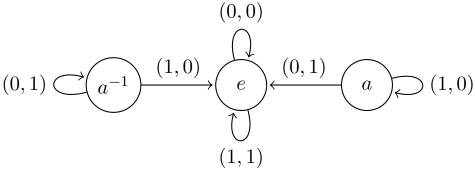

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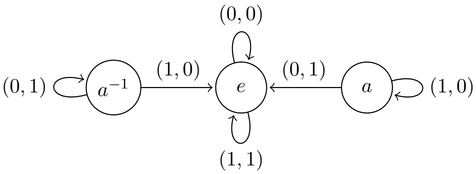

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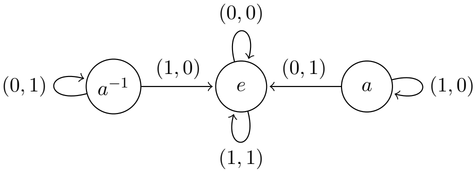

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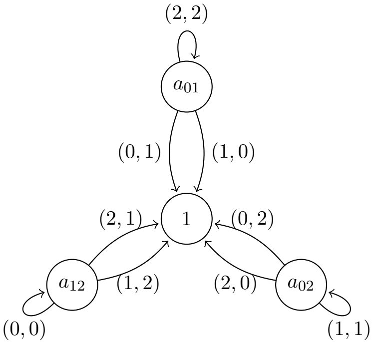
- Left-infinite paths define an *asymptotic equivalence* relation $\sim_G$ on $\mathsf{X}^{-\omega}$; $\mathcal{J}_G = \mathsf{X}^{-\omega}/\sim_G$ is the *limit space* of G
- Binary adding machine: (read from the right)

$$\overline{0}1w \sim_G \overline{1}0w$$

  Limit space is the circle

Hanoi Towers Group and Sierpiński Gasket

- Hanoi Towers Group:



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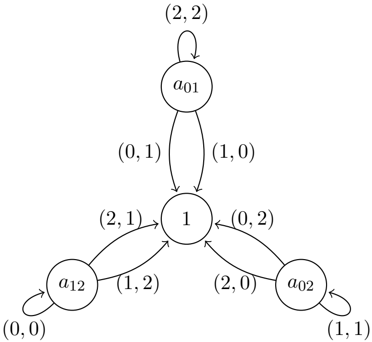

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- Limit space: Sierpiński Gasket

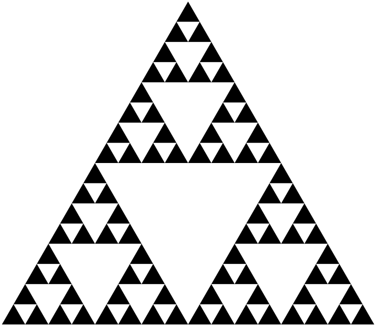

Figure: Sierpiński Gakset

Self-Similar Structures (Kigami)

- $F_i : K \to K$ continuous injection for each $i \in \mathsf{X}$, mapping K to a smaller part of itself
- A surjection $\pi : \mathsf{X}^{-\omega} \to K$ from the code space $\mathsf{X}^{-\omega}$ to K, marking the image of F_i by i
- $\mathcal{L} = (K, \mathsf{X}, \{F_i\}_{i \in \mathsf{X}})$ is a *self-similar structure on* K
- For a point $a \in K$, $\pi^{-1}(a)$ contains the "addresses" of a
- Example: Sierpiński Gasket (Usual Structure)

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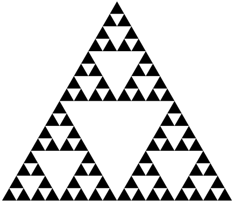

Figure: Sierpiński Gasket

When Does a Limit Space Have a Self-Similar Structure?

- A self-similar structure $\mathcal{L} = (\mathcal{J}_G, \mathsf{X}, \{F_i\}_{i \in \mathsf{X}})$ on a limit space $\mathcal{J}_G$, such that $p = \pi$
- Limit space of the binary adding machine, the circle, does not have a self-similar structure

Theorem (1)

The limit space $\mathcal{J}_G$ has a self-similar structure if and only if it satisfies the following condition:
For every left-infinite path $e = \ldots e_2 e_1$ in the nucleus ending at a non-trivial state and for every $w \in \mathsf{X}^*$, there exists a left-infinite path $f = \ldots f_2 f_1$ in the nucleus ending at a state g, such that the label of the edge f_n is the same as the label of e_n, and $g(w) = w$.

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When Does a Limit Space Have a P.C.F. Self-Similar Structure? (Slide I)

- Limit space: *finitely ramified in the group-theoretical sense* if the intersection of distinct tiles of the same level is finite
- Self-similar structure:
  - *finitely ramified in the fractal sense* if the intersection of the images of F_i is finite
  - *post-critically finite (p.c.f.)* if
    1. the set of addresses of the intersection of F_i is finite
    2. each address in this set has a recurring tail

  Significance: Can define Laplacian on the space
- (Bondarenko and Nekrashevych) A contracting group G is *p.c.f.* if there exists a finite number of left-infinite paths in its nucleus that end at a non-trivial state

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Lemma (Bondarenko and Nekrashevych 2003)

The limit space $\mathcal{J}_G$ is finitely ramified in the group-theoretical sense if and only if G is p.c.f.

Theorem (2)

The self-similar structure $\mathcal{L} = (\mathcal{J}_G, \mathsf{X}, \{F_i\}_{i \in \mathsf{X}})$ on the limit space $\mathcal{J}_G$ of a contracting G is p.c.f. if and only if G is p.c.f.

- Point 1: Finitely ramified in the group-theoretical sense is the same as in the fractal sense when $\mathcal{J}_G$ has a self-similar structure
- Point 2: Justifies use of the term “p.c.f. group”

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When Does a Limit Space have a P.C.F. Self-Similar Structure? (Slide III)

Corollary

A limit space $\mathcal{J}_G$ has a p.c.f. self-similar structure if and only if G satisfies the condition in Theorem (1) and is p.c.f.

Corollary

The self-similar structure on a limit space is p.c.f. if and only if it is finitely ramified.

- Example: The Kameyama fractal is not a limit space.

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Motivation

- A contracting group produces a limit space, which may have a self-similar structure
- Question: Given a self-similar structure, can we find a contracting group that produces a limit space with this structure? When?
- Focus: P.c.f. self-similar structures
- Necessary condition: If $\pi(\ldots x_2 x_1) = \pi(\ldots y_2 y_1)$, then $\pi(\ldots x_{n+1} x_n) = \pi(\ldots y_{n+1} y_n)$ for all n
- Equivalently: the *induced shift map* $\mathsf{s} : \mathcal{J}_G \to \mathcal{J}_G$ defined by $\mathsf{s} = F_i^{-1}$ for each image of F_i, exists and is continuous
- Why? It has to be a limit space!

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Construction (Slide I)

- For a p.c.f. self-similar structure $\mathcal{L}$ satisfying the necessary condition, $\pi(\ldots x_2 x_1) = \pi(\ldots y_2 y_1)$ implies that
  1. $\pi(\ldots x_{n+1} x_n) = \pi(\ldots y_{n+1} y_n)$ for all $n \in \mathbb{Z}^+$
  2. $\pi(\ldots x_2 x_1 w) = \pi(\ldots y_2 y_1)$ for all $w \in \mathsf{X}^*$
- Write down the "equivalence classes" induced by $\mathcal{L}$ systematically:
  - By (1) and (2), $\pi(\ldots x_{N+1} x_N w) = \pi(\ldots y_{N+1} y_N w)$ where $x_N \neq y_N$, which accounts for the original equation. Therefore, assume $x_1 \neq y_1$.
  - Then $\ldots x_2 x_1, \ldots y_2 y_1 \in \mathcal{C}$. $\mathcal{C}$ is finite, so there are only finitely many such equations.
  - $\mathcal{L}$ p.c.f. implies that elements in $\mathcal{C}$ have a recurring tail, so we can write $\pi(\bar{z} x_n \ldots x_2 x_1 w) = \pi(\bar{z} y_n \ldots y_2 y_1 w)$. Also, $z_k \neq x_n$ or y_n, and z is the shortest recurring word.

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- Write down the "equivalence classes" induced by $\mathcal{L}$ systematically (continued):
  - By (2) and induction, if $\pi(\overline{z}x_n \dots x_2 x_1 w) = \pi(\overline{z}y_n \dots y_2 y_1 w)$, then $\pi(\overline{z}\xi_n \dots \xi_2 \xi_1 w) = \pi(\overline{z}x_n \dots x_2 x_1 w)$ whenever $\xi_j \in \{x_j, y_j\}$ for all j.
  - Therefore, we can write all equivalence classes in the form
    $$\{\overline{z}\zeta_n \dots \zeta_2 \zeta_1 w \mid z, w \in \mathsf{X}^*, \zeta_j \in S_j\}$$
    for fixed $W, z \in \mathsf{X}^*$ and some $S_j \in \mathsf{X}$; we introduce a shorthand to denote this:
    $$\overline{z}S_n \dots S_2 S_1 w.$$
  - Notice that each equivalence class is determined by
    $$\overline{z}S_n \dots S_2 S_1 = \pi^{-1}(\alpha)$$
    for some $\alpha \in \pi(\mathcal{C})$, where $|S_1| > 1$. We use α to label $\overline{z}S_n \dots S_2 S_1$.

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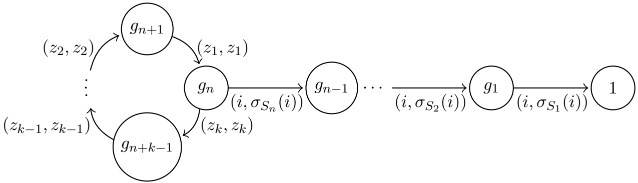

Figure: The generators corresponding to $\alpha = \Psi(\overline{z}S_n \ldots S_2S_1w)$

- The desired group $G_{\mathcal{L}}$ is the group generated by all the generators defined above for *all* $\alpha \in \pi(\mathcal{C})$

Theorem (3)

$G_{\mathcal{L}}$ produces a self-similar structure $\mathcal{L}'$ that is isomorphic to $\mathcal{L}$.

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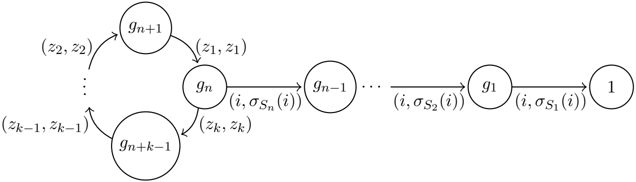

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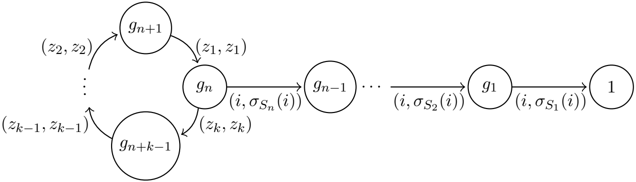

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Theorem (3)

$G_{\mathcal{L}}$ produces a self-similar structure $\mathcal{L}'$ that is isomorphic to $\mathcal{L}$.

Example: Unit Interval

- For the usual self-similar structure, the induced shift map s does not exist!
- A twisted structure:

$$I = [0,1]$$

$$F_0(x) = -(1/2)x + 1/2,\ F_1(x) = (1/2)x + 1/2$$

- All equivalence classes determined by the equivalence class

$$\pi^{-1}(1/2) = \bar{1}S_2S_1,$$

where $S_2 = \{0\}$ and $S_1 = \{0,1\}$, i.e. $\bar{1}00w \sim \bar{1}01w$.

- We define the group as follows:

Figure: The generators of $G_{\mathcal{L}}$

- Compare with the Grigorchuk group

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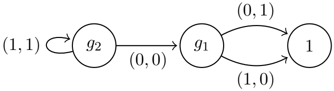

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Example: Sierpiński Gasket

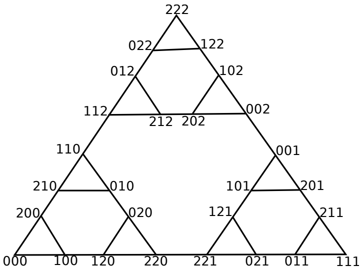

Figure: Sierpiński Gasket (Twisted Structure)

- s does not exist for usual structure; need a “twisted” structure
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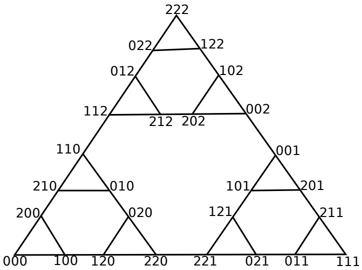

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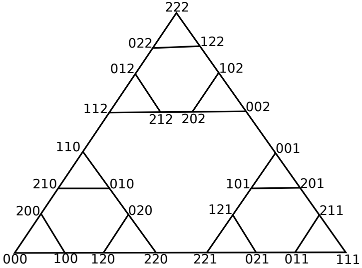

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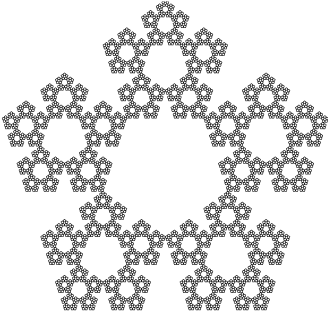

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- s does exist, so our construction yields a group

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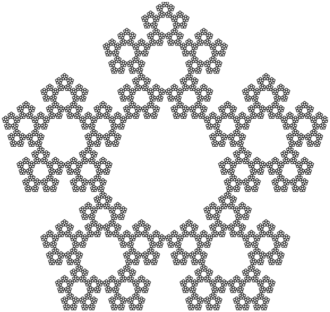

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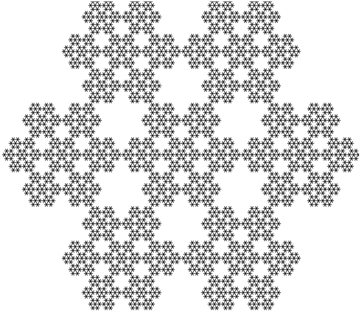

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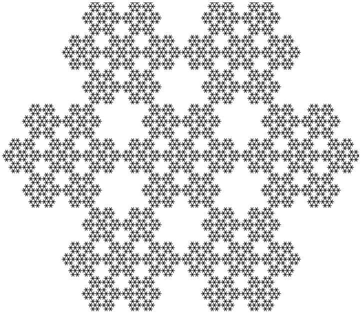

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Summary

- We clarified the condition for a limit space to have a self-similar structure.
- We clarified the condition for a self-similar structure on a limit space to be p.c.f.
- We constructed a contracting group that produces a given p.c.f. self-similar structure, and determined the necessary and sufficient condition for this inverse problem to have a solution.

- Outlook:
  - The inverse problem for non-p.c.f. self-similar structures

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