Ribbon homology cobordisms

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Tech Topology Conference 2019

Ribbon cobordisms

- For compact 3-manifolds Y_- and Y_+ (with same ∂), a *cobordism*
$$W\colon Y_- \to Y_+$$
is made up of 1-, 2-, and 3-handles
- *Ribbon*: does not have 3-handles
- Natural examples: Stein cobordisms between contact 3-manifolds

Why "ribbon"?

- Ans: Related to *ribbon concordances* of knots in S^3, which are concordances with 0- and 1-handles, but no 2-handles

Observation

If $C\colon K_- \to K_+$ is a ribbon concordance, then the exterior

- $Y_\pm := S^3 \setminus K_\pm$
- $W := (S^3 \times [0,1]) \setminus C$

gives a ribbon homology cobordism $W\colon Y_- \to Y_+$.

- Here, homology cobordism means that the maps

$$H_*(Y_-) \to H_*(W) \leftarrow H_*(Y_+)$$

  induced by inclusion are isomorphisms.
- W, like C, has no topology in interior (detected by homology)

Fundamental groups

- $Y_\pm = S^3 \setminus K_\pm, \quad W = (S^3 \times [0,1]) \setminus C$

Theorem (Gordon 1981)

If $C\colon K_- \to K_+$ is a ribbon concordance, then

$$\pi_1(Y_-) \hookrightarrow \pi_1(W) \twoheadleftarrow \pi_1(Y_+).$$

Proof.

Uses the residual finiteness of knot groups $\pi_1(Y_\pm)$. □

Several decades later...

Observation

Geometrization (Perelman 2006) implies residual finiteness for closed 3*-manifold groups.*

Theorem (Gordon 1981)

If $W : Y_- \to Y_+$ *is a ribbon homology cobordism, then*

$$\pi_1(Y_-) \hookrightarrow \pi_1(W) \twoheadleftarrow \pi_1(Y_+).$$

- Roughly: $\pi_1(Y_-)$ is "no bigger" than $\pi_1(Y_+)$
- How can we use this?

Main results

Observation

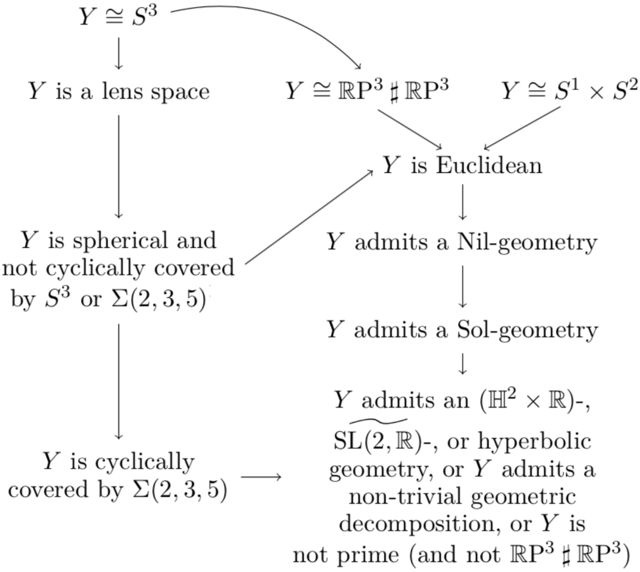
$\pi_1(Y)$ determines the Thurston geometry of Y (if it has one).

Theorem (Daemi–Lidman–Vela-Vick–W.)

If $W : Y_- \to Y_+$ is a ribbon homology cobordism, then

- *The Thurston geometries of Y_- and Y_+ satisfy a hierarchy.*

Ribbon homology cobordisms and Thurston geometries



Main results

Theorem (Daemi–Lidman–Vela-Vick–W.)

If $W : Y_- \to Y_+$ is a ribbon homology cobordism, then

- *The Thurston geometries of Y_- and Y_+ satisfy a hierarchy.*

- How else can we squeeze information from π_1?
- Idea: Representations of $\pi_1(Y_\pm)$

Main results

Theorem (Daemi–Lidman–Vela-Vick–W.)

If $W : Y_- \to Y_+$ is a ribbon homology cobordism, then

- *The Thurston geometries of Y_- and Y_+ satisfy a hierarchy.*
- *The dimension of the G-representation variety of Y_- is at most that of Y_+, for a compact Lie group G.*

- Any specific G? For example, $\mathrm{SU}(2)$
- Next idea: The $\mathrm{SU}(2)$-representations of $\pi_1(Y)$ are related to the instanton Floer homology $\mathrm{I}^\sharp(Y)$

Main results

Theorem (Daemi–Lidman–Vela-Vick–W.)

If $W : Y_- \to Y_+$ is a ribbon homology cobordism, then

- *The Thurston geometries of Y_- and Y_+ satisfy a hierarchy.*
- *The dimension of the G-representation variety of Y_- is at most that of Y_+.*
- $\mathrm{I}^\sharp(W) \colon \mathrm{I}^\sharp(Y_-) \to \mathrm{I}^\sharp(Y_+)$ *is injective.*

- Note: Conjecturally, $\mathrm{I}^\sharp(Y) \cong \widehat{\mathrm{HF}}(Y)$ (Heegaard Floer)
- Next idea: Similarly for Heegaard Floer homology!

Main results

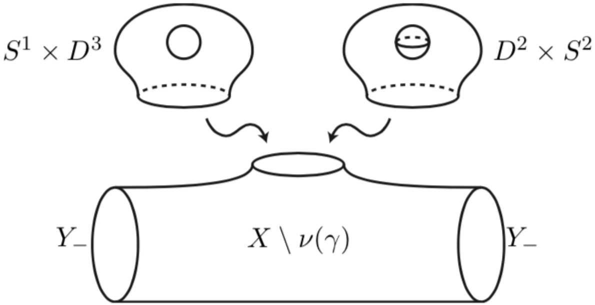
Theorem (Daemi–Lidman–Vela-Vick–W.)

If $W : Y_- \to Y_+$ is a ribbon homology cobordism, then

- *The Thurston geometries of Y_- and Y_+ satisfy a hierarchy.*
- *The dimension of the G-representation variety of Y_- is at most that of Y_+.*
- $\mathrm{I}^\sharp(W)\colon \mathrm{I}^\sharp(Y_-) \to \mathrm{I}^\sharp(Y_+)$ *is injective.*
- $\widehat{F}_W : \widehat{\mathrm{HF}}(Y_-) \to \widehat{\mathrm{HF}}(Y_+)$ *is injective.*

Sketch of proof for Floer homologies

- Doubling trick:



Attaching $S^1 \times D^3$ $\rightsquigarrow$ $X := (Y_- \times [0,1]) \sharp (S^1 \times S^3)$

Attaching $D^2 \times S^2$ $\rightsquigarrow$ $D(W) := W \cup_{Y_+} (-W)$

Application to Dehn surgery

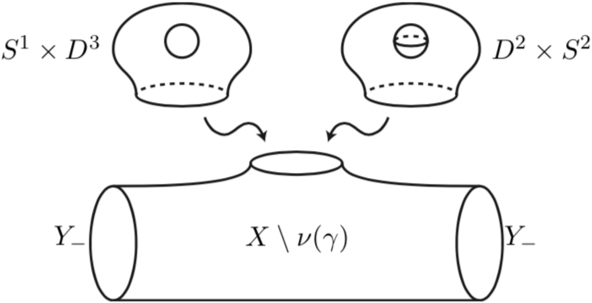
Theorem (Daemi–Lidman–Vela-Vick–W.)

Suppose that Y is a Seifert fibered homology sphere, K is a null-homotopic knot in Y, and $Y_0(K) \cong N \sharp (S^1 \times S^2)$. Then $N \cong Y$.

Proof.

Idea: A natural ribbon homology cobordism from N to Y. □

Thank you!



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