Upper bounds for the Lagrangian cobordism relation on Legendrian links

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Outline

1. Background
   - Lagrangian cobordisms between Legendrian links
2. Results
   - Upper bounds for the Lagrangian relation
   - The Lagrangian cobordism genus

Lagrangian cobordisms in $\mathbb{R} \times Y$

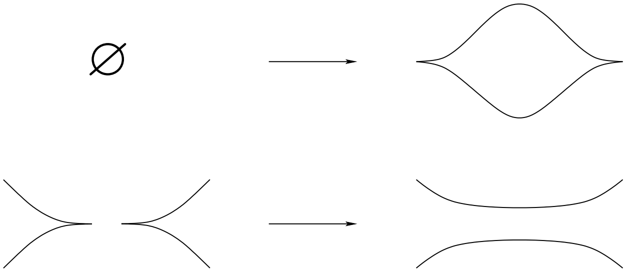
- Recall: Contact 3-manifold (Y, ξ), with $\xi = \ker(\alpha)$; e.g. $\mathbb{R}^3$ with standard contact structure $\xi_{\text{std}} = \ker(\alpha_{\text{std}})$, where $\alpha_{\text{std}} = dz - y\,dx$
- Tight vs. overtwisted contact manifolds
- A link $\Lambda \subset Y$ is *Legendrian* if Λ is tangent to ξ
- *Symplectization* $(\mathbb{R} \times Y, d(e^t\alpha))$
- (Chantraine) An *(exact) Lagrangian cobordism from Λ_- to Λ_+* is as follows:
  - An embedded Lagrangian submanifold $L \subset (\mathbb{R} \times Y, d(e^t\alpha))$
  - There is some real $T > 0$ such that

$$\mathcal{E}_+(L) := L \cap ((T, \infty) \times Y) = (T, \infty) \times \Lambda_+;$$
$$\mathcal{E}_-(L) := L \cap ((-\infty, -T) \times Y) = (-\infty, -T) \times \Lambda_-;$$

  - The rest of L is compact with boundary Λ_+ and $-\Lambda_-$
  - $e^t\alpha|_L = df$ is exact, with f constant on $\mathcal{E}_+(L) \cup \mathcal{E}_-(L)$

Decomposable Lagrangian cobordisms

- (Chantraine; Ekholm–Honda–Kálmán) A Lagrangian cobordism is *decomposable* if it is a product of elementary cobordisms: Legendrian isotopy, births, and saddles
- In front diagrams: Legendrian Reidemeister moves and



- Note the direction of arrows!

Decomposable Lagrangian cobordisms

- (Chantraine; Ekholm–Honda–Kálmán) A Lagrangian cobordism is *decomposable* if it is a product of elementary cobordisms: Legendrian isotopy, births, and saddles
- Every decomposable Lagrangian cobordism is exact
- Open question: Is every exact Lagrangian cobordism decomposable?

Some (perhaps unintuitive) properties

- Λ_- and Λ_+ Legendrian isotopic $\implies$ there exists a *Lagrangian concordance* (i.e. L is $\mathbb{R} \times S^1$)
- Lagrangian concordances are exact
- Differ from smooth cobordisms and concordances:
  - (Chantraine) If Λ_- and Λ_+ are knots in $\mathbb{R}^3$, then

$$r(\Lambda_+) = r(\Lambda_-), \quad tb(\Lambda_+) = tb(\Lambda_-) - \chi(L)$$

  - So a Lagrangian cobordism L cannot be inverted whenever $g(L) > 0$; can we invert Lagrangian concordances?
  - (Chantraine) Not in general!
  - So Lagrangian cobordism and concordance are not equivalence relations
- Instead, get a preorder $\preceq$ on Legendrian links
  - $\Lambda \preceq \Lambda$; $\Lambda_1 \preceq \Lambda_2$ and $\Lambda_2 \preceq \Lambda_3$ imply $\Lambda_1 \preceq \Lambda_3$
  - Questions: Is it a partial order? What can we say about the preorder?

Statement of main result

Theorem (with Sabloff and Vela-Vick)

Suppose that Λ and Λ' are oriented Legendrian links in a tight contact 3-manifold (Y, ξ), with the same rotation number (with respect to some Seifert surfaces). Then there exist Λ_- and Λ_+ such that $\Lambda_- \preceq \Lambda \preceq \Lambda_+$ and $\Lambda_- \preceq \Lambda' \preceq \Lambda_+$.

- Common lower bound established by Boranda, Traynor, and Yan

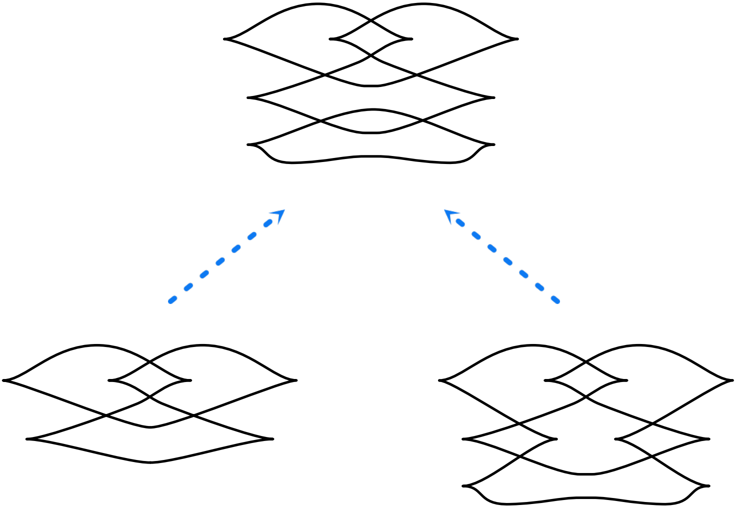
Theorem (with Sabloff and Vela-Vick)

With the same hypothesis, there exist $\Lambda_\pm$ and decomposable Lagrangian cobordisms L and L' from Λ_- to Λ_+ such that

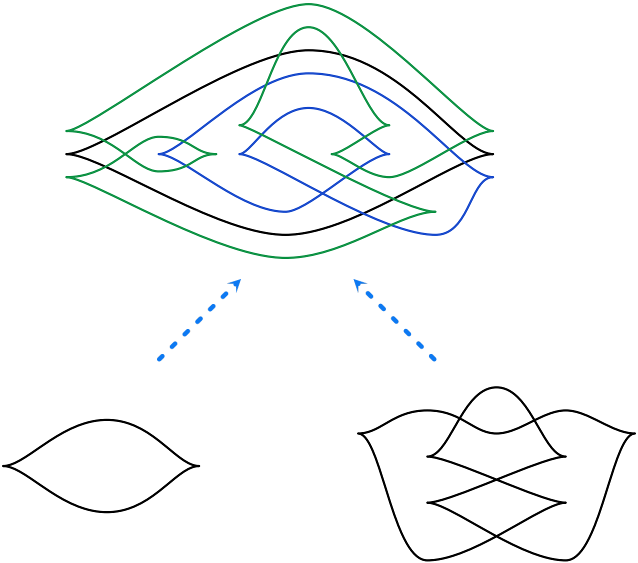
- *Λ appears as a collared slice of L, and Λ' appears as a collared slice of L'; and*
- *L and L' are exact-Lagrangian isotopic.*

- There is a version for unoriented Lagrangian cobordisms

Statement of main result



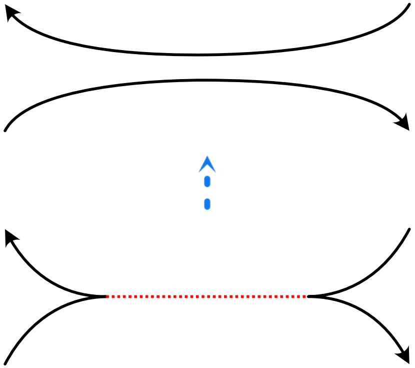
Statement of main result

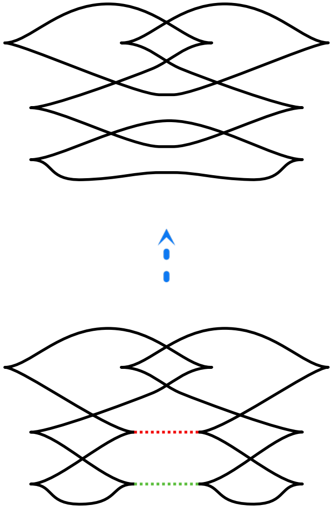


Sketch of proof

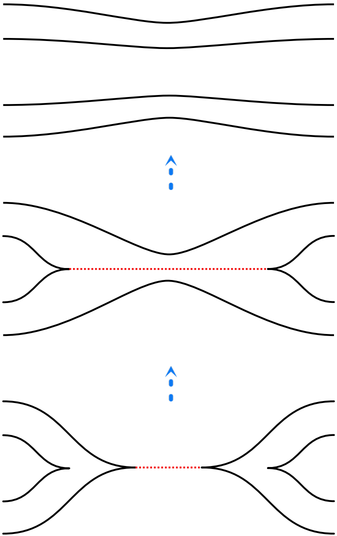
- Inspired by Lazarev's recent result for upper bounds of formally isotopic Legendrians in a contact $(2n+1)$-manifold with $n \geq 2$
- First, find a common lower bound Λ_- for Λ and Λ' by pinching
- Insight: Record the cobordisms from Λ_- to Λ and Λ' by a Legendrian graph
  - Legendrian ambient surgery (e.g. Dimitroglou Rizell) recovers cobordism
  - Caution: Cannot do this for all cobordisms
- Thus, need to make sure the cobordisms from Λ_- are recordable
  - In $\mathbb{R}^3$, refine the Boranda–Traynor–Yan diagrammatic approach
  - In general tight (Y, ξ), use convex surface theory
- Finally, combine the two Legendrian graphs to build both Λ_+ and the cobordisms to it

Sketch of proof





Sketch of proof



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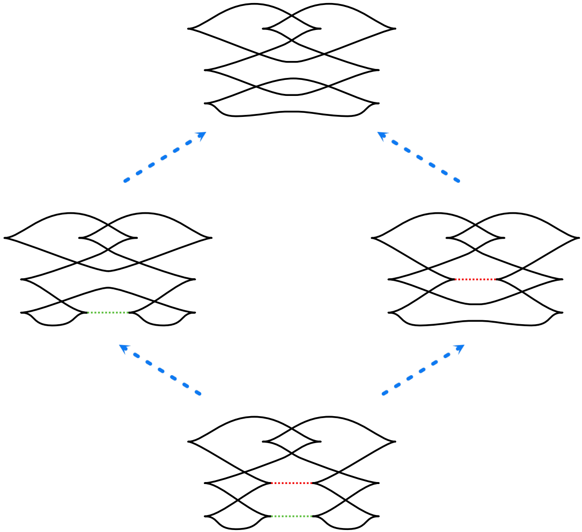
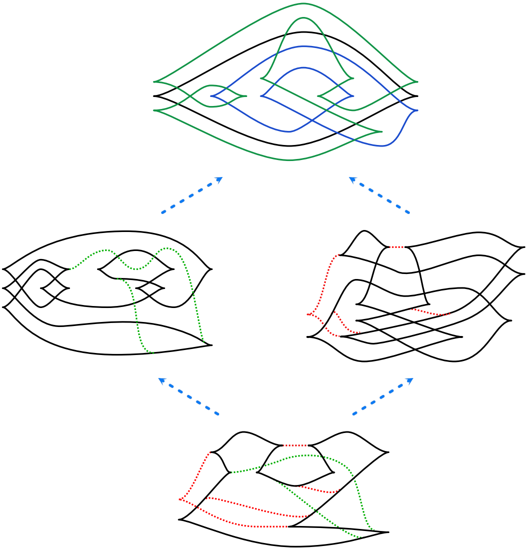


Sketch of proof



Lagrangian quasi-cobordisms

- Our result allows us to define the following:

Definition

A *Lagrangian quasi-cobordism* between Legendrian links Λ and Λ' consists of an ordered set of $n + 1$ Legendrian links

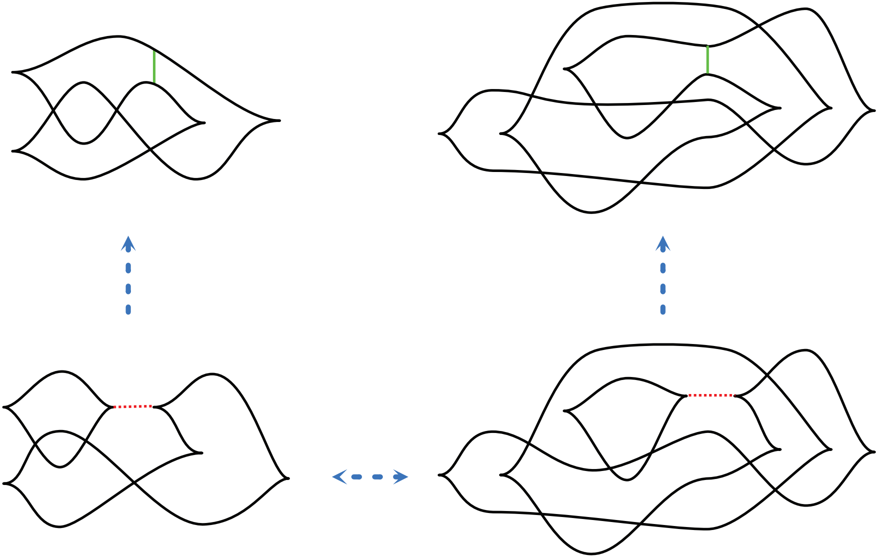
$$(\Lambda = \Lambda_0, \Lambda_1, \ldots, \Lambda_n = \Lambda')$$

an ordered set of n nontempty Legendrian links

$$(\Lambda_1^*, \ldots, \Lambda_n^*)$$

such that Λ_i^* is an upper or lower bound for the pair $(\Lambda_{i-1}, \Lambda_i)$, and all the Lagrangian cobordisms that realize these upper and lower bounds.

Lagrangian quasi-cobordisms



Lagrangian cobordism genus

Definition

The *Euler characteristic* of a Lagrangian quasi-cobordism is the sum of the Euler characteristics of the constituent Lagrangians, and its *genus* is computed from the Euler characteristic.
The *relative Lagrangian genus* $g_L(\Lambda, \Lambda')$ is the minimum genus of any Lagrangian quasi-cobordism between Λ and Λ'.

Corollary (with Sabloff and Vela-Vick)

Any two Legendrian links with the same rotation number are Lagrangian quasi-cobordant.

Basic properties

- $g_s(\Lambda, \Lambda') \leq g_L(\Lambda, \Lambda')$
- When is equality achieved?
- Not always, by considering double stabilizations
- But $g_s(\Lambda, \Lambda') = g_L(\Lambda, \Lambda')$ if Λ is Lagrangian fillable, and $\Lambda \preceq \Lambda'$
- Some open questions:
  - Is there a pair Λ and Λ' that are Lagrangian quasi-concordant but not Lagrangian concordant?
  - Can $g_L(\Lambda, \Lambda') - g_s(\Lambda, \Lambda')$ be arbitrarily large when Λ and Λ' both have maximal Thurston–Bennequin invariant?
  - Is there a version of this theory for Maslov-0 Lagrangians, which would allow the use of Legendrian contact homology?

Thank you!