S2500 ANALYSIS AND OPTIMIZATION, SUMMER 2016
FINAL EXAM SOLUTIONS

C.-M. MICHAEL WONG

No textbooks, notes or calculators allowed. The maximum score is 100. The number of points that each problem (and sub-problem) is worth is provided, so allocate your time well!

Problem 1 (10 points)**.** Consider the maximization problem

$$\max J(x), \quad J(x) = \int_{t_0}^{t_1} F(t, x, \dot{x})\, dt, \quad x(t_0) = x_0, \quad x(t_1) = x_1.$$

State, without proof, whether the following statements are true or false.

(a) (2 points) If $x^*(t)$ is an admissible function and satisfies the Euler–Lagrange Equation, then it is a solution to the maximization problem above.

(b) (2 points) If $x^*(t)$ is a solution to the maximization problem above, then it is a critical point of the functional $J(x)$.

(c) (2 points) If there exists a solution to the maximization problem above, then the function $F(t, x, \dot{x})$ must be concave in $(x, \dot{x})$.

(d) (2 points) If the function $F(t, x, \dot{x})$ is concave in $(x, \dot{x})$, then there exists a solution to the maximization problem above.

(e) (2 points) If $F(t, x, \dot{x}) = (-x^2 - 4x\dot{x} - \dot{x}^2)e^{-t}$, then $F(t, x, \dot{x})$ is concave in $(x, \dot{x})$ for all $t \in [t_0, t_1]$.

Solution. The answers are:

(a) False. The function $x^*(t)$ is only known to be a *candidate* solution in this case.

(b) True. This is similar to the case in vector calculus: If $x \in \mathbb{R}^n$ is a solution to the maximization problem $\max f(x)$ (with no constraints), then x is a critical point of the function f.

(c) False. The concavity of F is a sufficient condition for some function $x^*(t)$ that satisfies the Euler–Lagrange Equation to be a solution, but it is not a necessary condition.

(d) False. For example, for some ("unreasonable") initial and terminal conditions, the solutions to the Euler–Lagrange Equation may not be admissible.

(e) False. Even though F is concave in x and concave in $\dot{x}$, this does not imply that F is concave in $(x, \dot{x})$. In fact,

$$\operatorname{Hess}_{(x,\dot{x})} F = e^{-t} \begin{pmatrix} -2 & -4 \\ -4 & -2 \end{pmatrix}$$

has eigenvalues $-6e^{-t}$ and $2e^{-t}$, and hence is indefinite.

This concludes the problem. □

Problem 2 (10 points)**.** This problem is about the theory of calculus of variations. For Part (a) to Part (c), remember that the 13th century was the 1200s, for example. Due to the nature of the problem, you will receive full credit in Parts (f) to (h) as long as you describe the *idea* accurately, without any technical details. Please do not write more than a sentence or two for each part.

(a) (1 points) State the century in which the calculus of variations was first developed.
(b) (1 points) State the century in which the Euler–Lagrange Equation was developed.
(c) (1 points) State the century in which control theory was developed.
(d) (1 points) State the last name of any one of the few mathematicians who first solved the Brachistochrone Problem. (They were the pioneers in the theory of calculus of variations.)
(e) (1 points) State the last name of the mathematician associated to the Maximum Principle.
(f) (2 points) State, but do not solve, either the Brachistochrone Problem or the Isoperimetric Problem. What is the final conclusion that can be reached using calculus of variations?
(g) (2 points) State Hamilton's Principle of Least Action.
(h) (1 points) State the Fundamental Lemma of the Calculus of Variations.

Solution. The answers are:

(a) 17th century. The Brachistochrone Problem was posed by Johann Bernoulli in 1696.
(b) 18th century. Euler and Lagrange's work was done in the 1750s.
(c) 20th century. The Maximum Principle was formulated in the 1950s.
(d) The mathematicians were Issac Newton, Johann Bernoulli, Jakob Bernoulli, Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus and Guillaume de l'Hôpital.
(e) The Maximum Principle was formulated by Lev Pontryagin and his students.
(f) The Brachistochrone Problem is the problem of finding the curve along which a particle at rest descends without friction in the shortest time, given the initial and terminal points. The answer is a cycloid.
The Isoperimetric Problem is the problem of finding the smooth closed curve in the plane with a given length that encloses the maximum area. The answer is a circle.
(g) Hamilton's Principle of Least Action states that, in a system described by $\mathbf{r}(t)$, with initial state $\mathbf{r}_0 = \mathbf{r}(t_0)$ and terminal state $\mathbf{r}_1 = \mathbf{r}(t_1)$, the function $\mathbf{r}(t)$ is a critical point of the action functional

$$\mathcal{A}(\mathbf{r}) = \int_{t_0}^{t_1} L(t, \mathbf{r}, \dot{\mathbf{r}})\, dt,$$

where $L(t, \mathbf{r}, \dot{\mathbf{r}})$ is the Lagrangian function, defined by $L = K - U$, where K is the kinetic energy and U is the potential energy of the system.
(h) The Fundamental Lemma of the Calculus of Variations states that, if $f \in \mathcal{C}^0([t_0, t_1])$ satisfies

$$\int_{t_0}^{t_1} f(t)\mu(t)\, dt = 0$$

for every $\mu \in \mathcal{C}^2([t_0, t_1])$ that satisfies $\mu(t_0) = \mu(t_1) = 0$, then $f(t) = 0$ for all $t \in [t_0, t_1]$.

This concludes the problem. □

Problem 3 (20 points)**.** Solve the problem

$$\min J(x), \quad J(x) = \int_0^\pi \dot{x}^2\, dt, \quad x(0) = 2, \quad x(\pi) \geq \pi + 2,$$

subject to the constraint

$$\int_0^\pi \dot{x} \cos t\, dt = \pi.$$

You do not need to compute the actual minimum value. Make sure to state and justify all conditions you use. *Hint:* $\cos^2 t = (1 + \cos 2t)/2$.

Solution. If x^* is an optimal solution to the problem, then x^* must be a critical point of the functional

$$J(x) = \int_0^\pi (\dot{x}^2 - \lambda \dot{x} \cos t)\, dt$$

for some value of λ. The integrand is

$$F(t, x, \dot{x}) = \dot{x}^2 - \lambda \dot{x} \cos t,$$

so we see that

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial \dot{x}} = 2\dot{x} - \lambda \cos t.$$

The Euler–Lagrange Equation becomes

$$0 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 - 2\ddot{x} - \lambda \sin t,$$

which implies that

$$\ddot{x}^* = -\frac{\lambda}{2}\sin t, \quad \dot{x}^* = \frac{\lambda}{2}\cos t + C, \quad x^* = \frac{\lambda}{2}\sin t + Ct + D.$$

Now $x(0) = 2$ implies that $D = 2$. Moreover, we can evaluate

$$\pi = \int_0^\pi \dot{x} \cos t\, dt = \int_0^\pi \left(\frac{\lambda}{2}\cos^2 t + C\cos t\right) dt = \left(\frac{\lambda}{2}\left(\frac{t}{2} + \frac{\sin 2t}{4}\right) + C \sin t\right)\Big|_0^\pi = \frac{\lambda\pi}{4},$$

and so $\lambda = 4$. This gives

$$x^* = 2\sin t + Ct + 2, \quad \dot{x}^* = 2\cos t + C.$$

The transversality condition for $x(\pi) \geq \pi + 2$ is as follows. If $x^*(\pi) > \pi + 2$, then

$$0 = \left(\frac{\partial F^*}{\partial \dot{x}}\right)\Big|_{t=\pi} = 2\dot{x}^*(\pi) - \lambda \cos \pi = (4\cos\pi + 2C) - 4\cos\pi = 2C,$$

so $C = 0$. But then $x^*(\pi) = 0 + 0 + 2 \not> \pi + 2$, and so this case is rejected.

If, instead, $x^*(\pi) = \pi + 2$, then $C = 1$. In this case, we need to check that

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)\Big|_{t=\pi} = 2\dot{x}^*(\pi) - 4\cos\pi = 2C = 2 \geq 0.$$

Thus, $x^*(t) = 2\sin t + t + 2$ is the only admissible function that satisfies the necessary conditions.

Since F is the sum of a convex function and an affine function, we see that x^* is indeed the optimal solution. $\square$

Problem 4 (16 points)**.** Consider the problem

$$\min J(x), \quad J(x) = \int_{t_0}^{t_1} x\sqrt{1 + \dot{x}^2}\, dt, \quad x(t_0) = x_0, \quad x(t_1) = x_1.$$

Observe that the integrand $F(t, x, \dot{x}) = F(x, \dot{x})$; i.e. it does not explicitly depend on t.

(a) (10 points) Write down the *first integral* of the Euler–Lagrange Equation for the optimization problem above. Do not solve. *Hint: You should be writing down an equation.*

(b) (6 points) Does the first integral in Part (a) imply the Euler–Lagrange Equation? If not, what other condition is necessary? Check the condition for the optimization problem above.

Remark (0 points)**.** What shape does a hanging cable supported only at its ends assume under its own weight? Suppose that the shape of the cable is described by the graph of $x = x(t)$, and that the ends are at $(t, x) = (t_0, x_0)$ and (t_1, x_1). Then $J(x)$ in Problem 4 computes the height of its center of mass, which has to be minimized. Solving the optimization problem in Problem 4 (which you are not asked to do) gives the solution

$$x^*(t) = C \cosh\left(\frac{t + D}{C}\right),$$

where C and D are constants. This is the classical *catenary problem.*

The first integral of the Euler–Lagrange Equation is also called the *Beltrami Identity.* (This name was not mentioned in class.)

Solution. The solutions are as follows.

(a) The Beltrami Identity (i.e. the first integral) states that

$$F - \dot{x}\frac{\partial F}{\partial \dot{x}} = C,$$

where C is a constant. The integrand is

$$F(x, \dot{x}) = x\sqrt{1 + \dot{x}^2},$$

so we have

$$\frac{\partial F}{\partial \dot{x}} = \frac{x\dot{x}}{\sqrt{1 + \dot{x}^2}}.$$

Thus, the Beltrami Identity is

$$x\sqrt{1 + \dot{x}^2} - \frac{x\dot{x}^2}{\sqrt{1 + \dot{x}^2}} = C.$$

(b) The Beltrami Identity does not imply the Euler–Lagrange Equation. A sufficient condition for it to imply the Euler–Lagrange Equation is that

$$\frac{\partial F}{\partial x}(x, 0) \neq 0$$

for all x. In the present context,

$$\frac{\partial F}{\partial x} = \sqrt{1 + \dot{x}^2},$$

and so the sufficient condition reduces to

$$0 \neq \frac{\partial F}{\partial x}(x, 0) = \sqrt{1 + 0^2} = 1,$$

which holds.

This concludes the problem. □

Problem 5 (20 points)**.** Recall that in the Ramsey optimal savings problem

$$\max \int_0^T (U(f(K)) - \dot{K})e^{-rt}\,dt, \quad K(0) = K_0, \quad K(T) = K_T,$$

where $f'(K) > 0, f''(K) \leq 0, U'(C) > 0$ and $U''(C) < 0$, any solution to the differential equation

$$\ddot{K} - f'(K)\dot{K} + \frac{U'(C)}{U''(C)}(r - f'(K)) = 0,$$

where $C = f(K) - \dot{K}$, that satisfies the boundary conditions, must be a solution to the optimization problem. (Here, K is the capital, f is the productivity of capital, C is the consumption, U is the utility and r is the discount rate.) Use the differential equation to solve the problem

$$\max \int_0^T \ln(3K - \dot{K})e^{-2t}\, dt, \quad K(0) = K_0, \quad K(T) = K_T.$$

You do not need to compute the actual maximum value. *Hint: You only need to check the first- and second-order conditions on f and U and solve the given differential equation, since these already imply the necessary and sufficient conditions.*

Solution. Observe that $f(K) = 3K$, $U(C) = \ln C$ and $r = 2$. This means that

$$f'(K) = 3, \quad U'(C) = \frac{1}{C}, \quad U''(C) = -\frac{1}{C^2}, \quad \frac{U'(C)}{U''(C)} = -C.$$

Clearly, the first- and second-order conditions on f and U are satisfied. The given differential equation reduces to

$$0 = \ddot{K} - 3\dot{K} - C(2-3) = \ddot{K} - 3\dot{K} + (3K - \dot{K}) = \ddot{K} - 4\dot{K} + 3K$$

The characteristic equation is $\rho^2 - 4\rho + 3 = 0$, which immediately yields the roots $\rho = 3$ and $\rho = 1$. Therefore, the general solution to the differential equation is

$$K = Ae^{3t} + Be^t.$$

The initial condition $K(0) = K_0$ and the terminal condition $K(T) = K_T$ imply

$$A + B = K_0, \quad Ae^{3T} + Be^T = K_T,$$

which lead to

$$A = \frac{K_T - K_0e^T}{e^{3T} - e^T}, \quad B = \frac{K_0e^{3T} - K_T}{e^{3T} - e^T}.$$

Thus,

$$K^*(t) = \frac{K_T - K_0e^T}{e^{3T} - e^T}e^{3t} + \frac{K_0e^{3T} - K_T}{e^{3T} - e^T}e^t$$

is the only solution to the optimization problem. □

Problem 6 (24 points)**.** Solve the control problem

$$\max_{u(t)\in(-\infty,\infty)} \int_0^1 (-x^2 - u^2)\, dt, \quad \dot{x}(t) = -\frac{625}{2}t^2 + 3x + 4u, \quad x(0) = 1, \quad x(1) \text{ free.}$$

You do not need to compute the actual maximum value. Make sure to state and justify all conditions you use. Since the constants of integration are tedious to find, you do **NOT** need to find the constants. However, you must write down the transversality condition carefully. *Hint: You will have to compute $\ddot{p}$ at some point. Also, express x in terms of p and $\dot{p}$.*

Solution. The Hamiltonian is

$$H(t, x, u, p) = -x^2 - u^2 + p\left(-\frac{625}{2}t^2 + 3x + 4u\right),$$

so we see that

$$\frac{\partial H}{\partial x} = -2x + 3p, \quad \frac{\partial H}{\partial u} = -2u + 4p.$$

The Maximum Principle states that if (x^*, u^*) is an optimal pair, then there exists a continuous function p such that for each $t \in [0, \pi]$,

$$u^*(t) \text{ maximizes } H(t, x^*(t), u, p(t)) \text{ for } u \in (-\infty, \infty), \text{ and} \tag{1}$$

$$\dot{p} = -\frac{\partial H}{\partial x}(t, x^*, u^*, p) = 2x^* - 3p. \tag{2}$$

Here, (2) implies that

$$x^* = \frac{1}{2}\dot{p} + \frac{3}{2}p. \tag{3}$$

Since H is concave in u (because it is a sum of concave functions) and the control region is $(-\infty, \infty) = \mathbb{R}$, (1) is equivalent to the following condition:

$$\frac{\partial H}{\partial u}(t, x^*, u^*, p) = -2u^* + 4p = 0,$$

which implies that

$$u^* = 2p. \tag{4}$$

Following the hint, we differentiate (2) to get

$$\ddot{p} = 2\dot{x}^* - 3\dot{p}.$$

Using the fact that $\dot{x}^* = -625t^2/2 + 3x^* + 4u^*$, we obtain

$$\ddot{p} = 2\left(-\frac{625}{2}t^2 + 3x^* + 4u^*\right) - 3\dot{p} = -625t^2 + 6x^* + 8u^* - 3\dot{p}.$$

We now use (3) and (4) to expand this:

$$\ddot{p} = -625t^2 + 6\left(\frac{1}{2}\dot{p} + \frac{3}{2}p\right) + 16p - 3\dot{p} = -625t^2 + 25p,$$

which we put into the standard form

$$\ddot{p} - 25p = -625t^2. \tag{5}$$

This is a non-homogeneous second-order linear ordinary differential equation. We first find the general solution for the homogeneous equation

$$\ddot{p} - 25p = 0. \tag{6}$$

Its characteristic equation is $r^2 - 25 = 0$, which gives us $r = \pm 5$. Thus, the general solution for the homogeneous equation (6) is

$$p = Ae^{5t} + Be^{-5t}.$$

Now we look for a particular solution to (5). We guess that it is a polynomial of degree 2, where $p = at^2 + bt + c$; then $\dot{p} = 2at + b$ and $\ddot{p} = 2a$. So we obtain

$$-625t^2 = \ddot{p} - 25p = 2a - 25(at^2 + bt + c) = -25at^2 - 25bt + (2a - 25c).$$

This implies that $a = 25, b = 0, c = 2$, and so our particular solution is $p = 25t^2 + 2$. Thus, the general solution to (5) is

$$p = Ae^{5t} + Be^{-5t} + 25t^2 + 2.$$

The transversality condition on p for $x(1)$ being free is that $p(1) = 0$. This implies

$$Ae^5 + Be^{-5} = -27. \tag{7}$$

Now
$$\dot{p} = 5Ae^{5t} - 5Be^{-5} + 50t,$$
so using (3), we obtain
$$x^* = \frac{1}{2}\dot{p} + \frac{3}{2}p = \frac{1}{2}(8Ae^{5t} - 2Be^{-5t} + 75t^2 + 50t + 6) = 4Ae^{5t} - Be^{-5t} + \frac{75}{2}t^2 + 25t + 3.$$
Using (4), we also get
$$u^* = 2p = 2Ae^{5t} + 2Be^{-5t} + 50t^2 + 4.$$
The condition $x(0) = 1$ now gives us
$$1 = x^*(0) = 4A - B + 3,$$
or simply
$$4A - B = -2. \tag{8}$$
Combining (7) and (8), we get
$$A(e^5 + 4e^{-5}) + 2e^{-5} = -27,$$
giving us
$$A = \frac{-27 - 2e^{-5}}{e^5 + 4e^{-5}}, \quad B = \frac{-108 + 2e^5}{e^5 + 4e^{-5}}.$$
We now conclude that
$$\begin{aligned} x^* &= \frac{-108 - 8e^{-5}}{e^5 + 4e^{-5}}e^{5t} + \frac{108 - 2e^5}{e^5 + 4e^{-5}}e^{-5t} + \frac{75}{2}t^2 + 25t + 3, \\ u^* &= \frac{-54 - 4e^{-5}}{e^5 + 4e^{-5}}e^{5t} + \frac{-216 + 4e^5}{e^5 + 4e^{-5}}e^{-5t} + 50t^2 + 4. \end{aligned}$$
Since H is concave in (x, u) (because it is a sum of concave functions), (x^*, u^*) is the unique optimal pair. $\square$