3.1

1. $f(x,y) = x^3 + y^3 - 3x - 2y \quad x,y>0$

$\nabla f = 3x^2 - 3,\ 3y^2 - 2$

$\nabla f = 0$ when $x = 1 \quad y = \sqrt{\frac{2}{3}}$

$f_{xx} = 6x \quad f_{yy} = 6y \quad f_{xy} = 0$

$f'' = \begin{matrix} 6x & 0 \\ 0 & 6y \end{matrix} \quad \begin{matrix} D_1 > 0 \text{ when } x>0 \\ D_2 > 0 \text{ when } x,y>0 \end{matrix}$

$\Rightarrow \boxed{\begin{matrix} f \text{ is strictly convex} \\ (1, \sqrt{\frac{2}{3}}) \text{ is is when } f \text{ reaches global minum value of } -2-\frac{4}{3}\sqrt{\frac{2}{3}} \end{matrix}}$

3a. max $\underbrace{pv_1^{1/3} v_2^{1/2}}_{\text{Cobb Douglas}} - q_1 v_1 - q_2 v_2$ so this is concave

$\frac{\partial \Pi}{\partial v_1} : \frac{\frac{1}{3} p v_1^{1/3} v_2^{1/2}}{v_1} = q_1 \qquad \frac{\partial \Pi}{\partial v_2} : \frac{\frac{1}{2} p v_1^{1/3} v_2^{1/2}}{v_2} = q_2$

$F = v_1^{1/3} v_2^{1/2} \qquad F = v_1^{1/3} v_2^{1/2}$

$3q_1 v_1 = pF \qquad 2q_2 v_2 = pF$

$v_1^* = \frac{pF}{3q_1} \qquad v_2^* = \frac{pF}{2q_2}$

$F = \left(\frac{pF}{3q_1}\right)^{1/3} \left(\frac{pF}{2q_2}\right)^{1/2} = (pF)^{5/6} \left(\frac{1}{3q_1}\right)^{1/3} \left(\frac{1}{2q_2}\right)^{1/2}$

$\left(F^{1/6} = p^{5/6} \left(\frac{1}{3q_1}\right)^{1/3} \left(\frac{1}{2q_2}\right)^{1/2}\right)^6$

$F = p^5 \left(\frac{1}{3q_1}\right)^2 \left(\frac{1}{2q_2}\right)^3$

$\boxed{v_1^* = \frac{p^6}{216 q_1^3 q_2^3} \qquad v_2^* = \frac{p^6}{144 q_1^2 q_2^4}}$

b. $\Pi^*(p, q_1, q_2) = \frac{p^6}{72 q_1^2 q_2^3} - \frac{\cancel{q_1} p^6}{216 q_1^{\cancel{3}2} q_2^3} - \frac{\cancel{q_2} p^6}{144 q_1^2 q_2^{\cancel{4}3}} = \frac{p^6 - \frac{1}{3}p^6 - \frac{1}{2}p^6}{72 q_1^2 q_2^3} = \frac{p^6}{432 q_1^2 q_2^3}$

$\frac{\partial \Pi^*}{\partial p} = \frac{6p^5}{432 q_1^2 q_2^3} = \frac{p^5}{72 q_1^2 q_2^3} \qquad \frac{\partial \Pi^*}{\partial q_1} = \frac{-2p^6}{432 q_1^3 q_2^3} = \frac{-p^6}{216 q_1^3 q_2^3} \qquad \frac{\partial \Pi^*}{\partial q_2} = \frac{-3p^6}{432 q_1^2 q_2^4} = \frac{-p^6}{144 q_1^2 q_2^4}$

$\frac{\partial \Pi}{\partial p}\Big|_{v=v^*} = v_1^{1/3} v_2^{1/2} = \frac{p^5}{72 q_1^2 q_2^3}$ same $\qquad \frac{\partial \Pi}{\partial q_1}\Big|_{v=v^*} = -v_1 = \frac{-p^6}{216 q_1^3 q_2^3}$ same $\qquad \frac{\partial \Pi}{\partial q_2}\Big|_{v=v^*} = -v_2 = \frac{-p^6}{144 q_1^2 q_2^4}$ same

$\frac{\partial \Pi}{\partial p}\Big|_{v=v^*} = \frac{\partial \Pi^*}{\partial p} = F^* \qquad \frac{\partial \Pi}{\partial q_1}\Big|_{v=v^*} = \frac{\partial \Pi^*}{\partial q_1} = -v_1^* \qquad \frac{\partial \Pi}{\partial q_2}\Big|_{v=v^*} = \frac{\partial \Pi^*}{\partial q_2} = -v_2^*$

6a. $F(v_1, \dots, v_n) = a_1 \ln(v_1+1) + \dots + a_n \ln(v_n+1)$

$$\Pi = pF - \sum_{i=1}^{n} q_i v_i$$

$$\frac{\partial \Pi}{\partial v_i} : \frac{pa_i}{v_i+1} = q_i \qquad pa_i - q_i$$

$$(v_i+1)q_i = pa_i$$

$$\boxed{v_i^* = \frac{pa_i}{q_i} - 1 \quad \text{for } i=1,\dots,n}$$ F concave so this is maximizer.

b. $$\Pi^* = p\underbrace{\left[a_1 \ln\left(\frac{pa_1}{q_1}\right) + \dots + a_n \ln\left(\frac{pa_n}{q_n}\right)\right]}_{F(v^*)} - \sum_{i=1}^{n}(pa_i - q_i) = \sum_{i=1}^{n}\left[pa_i \ln\left(\frac{pa_i}{q_i}\right) - pa_i + q_i\right]$$

$$= \sum_{i=1}^{n}\left[pa_i\left(\ln\left(\frac{pa_i}{q_i}\right) - 1\right) + q_i\right]$$

$$= \sum_{i=1}^{n}\left[pa_i \ln p + pa_i \ln a_i - pa_i \ln q_i - pa_i + q_i\right]$$

$$\frac{\partial \Pi^*}{\partial p} = \sum_{i=1}^{n}(a_i \ln p + \cancel{a_i} + a_i \ln a_i - a_i \ln q_i - \cancel{a_i})$$

$$= \sum_{i=1}^{n} a_i \ln \frac{pa_i}{q_i} = F(v^*)$$

$$\frac{\partial \Pi}{\partial p}\bigg|_{v=v^*} = \sum_{i=1}^{n} a_i \ln\left(\frac{pa_i}{q_i}\right) = F(v^*)$$

$$\frac{\partial \Pi^*}{\partial a_i} = \left[p \ln p + p \ln a_i + \cancel{p} - p \ln q_i - \cancel{p}\right]$$

$$= p \ln\left(\frac{pa_i}{q_i}\right)$$

$$\frac{\partial \Pi}{\partial a_i}\bigg|_{v=v^*} = p \ln(v_i^* + 1) = p \ln\left(\frac{pa_i}{q_i}\right)$$

$$\frac{\partial \Pi^*}{\partial q_i} = \frac{-pa_i}{q_i} + 1$$

$$\frac{\partial \Pi}{\partial q_i}\bigg|_{v=v^*} = -v_i = \frac{-pa_i}{q_i} + 1$$

5.2

1. $f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 3x_3^2 - x_1x_2 + 2x_1x_3 + x_2x_3$

$f_{x_1} = 2x_1 - x_2 + 2x_3 \qquad f_{x_1x_1} = 2 \qquad f_{x_1x_2} = -1$

$f_{x_2} = 2x_2 - x_1 + x_3 \qquad f_{x_2x_2} = 2 \qquad f_{x_2x_3} = 1$

$f_{x_3} = 6x_3 + 2x_1 + x_2 \qquad f_{x_3x_3} = 6 \qquad f_{x_3x_1} = 2$

$$\begin{matrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{matrix} \Rightarrow \begin{matrix} 1 & -\frac{1}{2} & 1 \\ 0 & \frac{3}{2} & 2 \\ 0 & 2 & 4 \end{matrix} \Rightarrow \begin{matrix} 1 & 0 & \frac{5}{3} \\ 0 & 1 & \frac{4}{3} \\ 0 & 0 & \frac{4}{3} \end{matrix} \Rightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

Stationary point: $(0,0,0)$

$f'' = \begin{bmatrix} 2 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 6 \end{bmatrix} \qquad \begin{aligned} D_1 &= 2 \\ D_2 &= 3 \\ D_3 &= 2(11) - (-1)(-8) + 2(-5) = 4 \end{aligned}$

$\Rightarrow f$ is strictly convex so $\boxed{(0,0,0) \text{ is a local and global minimum}}$

2a. $f(x,y) = x^3 + y^3 - 3xy$

$f_x = 3x^2 - 3y = 0 , \quad y = x^2 \qquad f_{xx} = 6x \qquad f_{xy} = -3$

$f_y = 3y^2 - 3x = 0 \qquad x = y^2 \qquad f_{yy} = 6y$

$3y^4 - 3y = 0$

$y(3y^3 - 3) = 0$

$y = 0, 1$

Stationary points: $(0,0), (1,1)$

From (9): $0 \cdot h_1^2 - 3h_1h_2 - 3h_2h_1 + 0 \cdot h_2^2 = \boxed{-6h_1h_2 \text{ at } (0,0)}$

$\boxed{6h_1^2 - 6h_1h_2 + 6h_2^2 \text{ at } (1,1)}$

b. $f'' = \begin{matrix} 6x & -3 \\ -3 & 6y \end{matrix}$ At $(0,0)$: $D_1 = 0$, $D_2 = -9$ $\Rightarrow$ indefinite

At $(1,1)$: $D_1 = 6$, $D_2 = 27$ $\Rightarrow$ positive definite

c. $(0,0)$ is a saddle point

$(1,1)$ is a local minimum

3a. $f(x,y,z) = x^2 + x^2y + y^2z + z^2 - 4z$

$f_x = 2x + 2xy \qquad f_{xx} = 2 + 2y \qquad f_{xy} = 2x$

$f_y = x^2 + 2yz \qquad f_{yy} = 2z \qquad f_{yz} = 2y$

$f_z = y^2 + 2z - 4 \qquad f_{zz} = 2 \qquad f_{zx} = 0$

a. continued

$0 = 2x(1+y)$ $\quad$ $0 = x^2 + 2yz$ $\quad$ $4 = y^2 + 2z$

$= x^2 + y(4-y^2)$ $\quad$ $z = \frac{4-y^2}{2}$

If $y = -1$, $z = \frac{3}{2}$, $x = \pm\sqrt{3}$ $\quad$ $x=0$, $y=0$, $z=2$ $\quad$ $x=0$, $y=+2$, $z=0$ $\quad$ $x=0$, $y=-2$, $z=0$

$$f'' = \begin{bmatrix} 2+2y & 2x & 0 \\ 2x & 2z & 2y \\ 0 & 2y & 2 \end{bmatrix}$$

At $(\sqrt{3}, -1, \frac{3}{2})$
$D_1 = 0$
$D_2 = -12$
$D_3 = -2\sqrt{3}(4\sqrt{3})$
$\boxed{\text{saddle point}}$

At $(-\sqrt{3}, -1, \frac{3}{2})$
$D_1 = 0$
$D_2 = -12$
$D_3 = -2\sqrt{3}(4\sqrt{3})$
$\boxed{\text{saddle point}}$

At $(0,0,2)$
$D_1 = 2$
$D_2 = 8$
$D_3 = 2(8)$
$\boxed{\text{local minimum}}$

At $(0,2,0)$
$D_1 = 6$
$D_2 = 0$
$D_3 = 6(-16)$
$\boxed{\text{saddle point}}$

At $(0,-2,0)$
$D_1 = -2$
$D_2 = 0$
$D_3 = -2(-16)$
$\boxed{\text{saddle point}}$

b. $f(x_1, x_2, x_3, x_4) = 20x_2 + 48x_3 + 6x_4 + 8x_1x_2 - 4x_1^2 - 12x_3^2 - x_4^2 - 4x_2^3$

$f_{x_1} = 8x_2 - 8x_1$ $\quad$ $f_{x_1x_1} = -8$ $\quad$ $f_{x_1x_2} = 8$ $\quad$ $f_{x_1x_3} = 0$

$f_{x_2} = 20 + 8x_1 - 12x_2^2$ $\quad$ $f_{x_2x_2} = -24x_2$ $\quad$ $f_{x_2x_3} = 0$ $\quad$ $f_{x_2x_4} = 0$

$f_{x_3} = 48 - 24x_3$ $\quad$ $f_{x_3x_3} = -24$ $\quad$ $f_{x_3x_4} = 0$

$f_{x_4} = 6 - 2x_4$ $\quad$ $f_{x_4x_4} = -2$ $\quad$ $f_{x_4x_1} = 0$

$x_4 = 3$ $\quad$ $x_1 = x_2$

$x_3 = 2$ $\quad$ $0 = 20 + 8x_2 - 12x_2^2$

$= -4(3x_2^2 - 2x_2 - 5)$

$= (3x_2 - 5)(x_2 + 1)$

$x_1 = x_2 = -1, \frac{5}{3}$

$$f'' = \begin{matrix} -8 & 8 & 0 & 0 \\ 8 & -24x_2 & 0 & 0 \\ 0 & 0 & -24 & 0 \\ 0 & 0 & 0 & -2 \end{matrix} \qquad -40$$

At $(-1,-1,2,3)$
$D_1 = -8$
$D_2 = -8(24) - 8(8) < 0$
$D_3 = +8(+24)(24) + 8(8)(24) > 0$
$D_4 = -8[24(24)(2)] - 8[8(48)] + 0 - 0 < 0$
$\boxed{(-1,-1,2,3) \text{ saddle point}}$

At $(\frac{5}{3}, \frac{5}{3}, 2, 3)$
$D_1 = -8 < 0$
$D_2 = 8(40) - 64 > 0$
$D_3 = -8(+40)(+24) + 8(8)(+24) < 0$
$D_4 = +8[+40(48)] - 8[8(48)] > 0$
$\boxed{(\frac{5}{3}, \frac{5}{3}, 2, 3) \text{ local maximum}}$

1. $f(x,y) = (1+y)^3 x^2 + y^2$

$f_x = 2x(1+y)^3$ $\quad f_{xx} = 2(1+y)^3$ $\quad f_{xy} = 6x(1+y)^2$

$f_y = 3(1+y)^2 x^2 + 2y$ $\quad f_{yy} = 6(1+y)x^2 + 2$

If $x = 0$

$y = 0$

Stationary point $(0,0)$

$$f'' = \begin{matrix} 2(1+y)^3 & 6x(1+y)^2 \\ 6x(1+y)^2 & 6(1+y)x^2+2 \end{matrix}$$

At $(0,0)$

$D_1 = 2$

$D_2 = 2(2) - 0$

$\Rightarrow (0,0)$ is a local minimum, $f(0,0) = 0$

Is it a global minimum?

When $(1-y)^3$ is negative, we can choose an arbitrarily large x to get $f(x,y)$ as small as we want.