2. a) max $\ln(x+1) + \ln(y+1)$ subject to $x + 2y \le c$
and $x + y \le 2$

So,

$$\mathcal{L} = \ln(x+1) + \ln(y+1) - \lambda_1(x+2y-c) - \lambda_2(x+y-2).$$

Thus,

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{1}{x+1} - \lambda_1 - \lambda_2 = 0$$

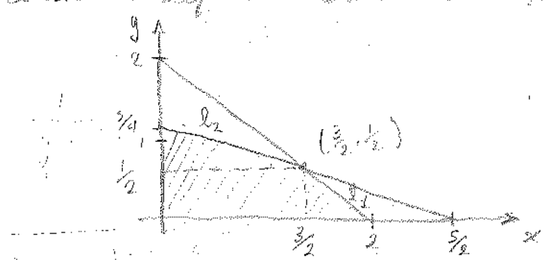
$$\frac{\partial \mathcal{L}}{\partial y} = \frac{1}{y+1} - 2\lambda_1 - \lambda_2 = 0$$

$\lambda_1 \ge 0$ with $\lambda_1 = 0$ if $x + 2y < c$

$\lambda_2 \ge 0$ with $\lambda_2 = 0$ if $x + y < 2$.

b) let $c = 5/2$

Since the constraints define a closed bdd region:



Since $f(x,y)$ is monotonically increasing, it is clear that the case where both constraints don't bind will not be a maximum. Thus, we eliminate the case $\lambda_1 = \lambda_2 = 0$. Additionally note there is only one point where both constraints bind is $(\frac{3}{2}, \frac{1}{2})$ where the value is $f(\frac{3}{2}, \frac{1}{2}) = \ln\left(\frac{15}{4}\right)$.

Thus, we only need to check the two other cases.

Suppose the first constraint doesn't bind. Then,

$\lambda_1 = 0$ and $x + 2y < \frac{5}{2}$ and $\lambda_2 \ge 0$ and $x + y = 2$.

This corresponds to the line l_2 in the graph. This implies

$$y + 1 = x + 1 \Rightarrow y = x$$

This implies $y = x = 1$. Which is not feasible since $3 > \frac{5}{2}$.

Now, let us check the second constraint.

So, $\lambda_1 \geq 0$ and $x + 2y = 5/2$ and $\lambda_2 = 0$ and $x + y < 2$ which implies

$$\frac{1}{x+1} = \frac{1}{2} \cdot \frac{1}{y+1},$$

$$\Rightarrow \quad 2y + 2 = x + 1$$

$$2y = x - 1$$

$$\Rightarrow \quad 2x = 7/2$$

$$\Rightarrow \quad x = 7/4$$

$$\Rightarrow \quad 2y = 3/4 \Rightarrow y = 3/8$$

$$\frac{14}{8} + \frac{3}{8} = \frac{17}{8} > 2.$$

Thus, this condition is not feasible and by the EVT we conclude that the maximum is $f^*(3/2, 1/2) = \ln\left(\frac{15}{4}\right)$ at $(x, y) = (3/2, 1/2)$.

c) Here $x + y = 2$ and $x + 2y = c$

$$\Rightarrow y = c - 2 \text{ and } x = 4 - c$$

And:

$$f^*(c) = \ln(5 - c) + \ln(c - 1)$$

$$f^{*\prime}(c) = \frac{1}{5-c} \cdot (-1) + \frac{1}{c-1} = \frac{1}{c-5} + \frac{1}{c-1} = \frac{c-1+c-5}{(c-5)(c-1)} = \frac{2c-6}{(c-5)(c-1)}$$

$$= \frac{-1}{\left(-\frac{5}{2}\right)\left(\frac{3}{2}\right)} = \frac{4}{15} \text{ at } c = 5/2.$$

3. This can be reformulated as:

$\max -4\ln(x^2 + 2) - y^2$ subject to $-x^2 - y \leq -2$, $-x \leq -1$.

Thus,

$$\mathcal{L} = -4\ln(x^2 + y) - y^2 - \lambda_1(-x^2 - y + 2) - \lambda_2(-x + 1)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{-4}{x^2 + y} \cdot 2x + 2\lambda_1 x + \lambda_2 = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{-4}{x^2 + y} - 2y + \lambda_1 = 0 \qquad (2)$$

$\lambda_1 \geq 0$ with $\lambda_1 = 0$ if $-x^2 - y \leq -2$

$\lambda_2 \geq 0$ with $\lambda_2 = 0$ if $-x \leq -1$

First multiply (2) by $-2x$ and add (1) & (2) to yield:

$$4xy + \lambda_2 = 0$$

Suppose that both constraints bind such that $\lambda_1, \lambda_2 \geq 0$ and

$$-x^2 - y = -2 \text{ and } -x = -1.$$

This implies that $x = 1$ and $y = 1$ which implies that $\lambda_2 = -4$.

Thus, there are no solutions when both constraints bind.

Now, suppose that the second constraint doesn't bind such that $\lambda_2 = 0$; this implies that either $x = 0$ or $y = 0$.

If $x = 0$, then $y = 2$ and if $y = 0$, then $x = \pm\sqrt{2}$.

For $(0,2)$ $\lambda_1 = 6$ and $\lambda_2 = 0$. For $(\pm\sqrt{2}, 0)$ $\lambda_1 = 2$ and $\lambda_2 = 0$.

But note $(0,2)$ is not a solution since $0 > -1$.

Thus, $(\pm\sqrt{2}, 0)$ $\lambda_1 = 2$, $\lambda_2 = 0$ is the only solution.

Now suppose the first constraint doesn't bind such that $\lambda_1 = 0$ and $\lambda_2 \geq 0$ $x = 1$. This implies:

$$-4 = 2y(1+y)$$

$$\Rightarrow 2y^2 + 2y + 4 = y^2 + y + 2 = 0$$

which has roots

$$y_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \cdot 2}}{2}$$ which has no real solutions

so there are no solutions for $\lambda_1 = 0$, $\lambda_2 \geq 0$.

Now suppose neither binds. So $\lambda_1 = \lambda_2 = 0$

Either $x = 0$ or $y = 0$ if $x = 0$ is $-4 = 2y^2$ which has no solutions. If $y = 0$ the K-T conditions are not satisfied.

Thus the only points are:

$$(\pm\sqrt{2}, 0, 2, 0).$$

$$f^*(+\sqrt{2}, 0) = 4\ln(4) = f^*(-\sqrt{2}, 0).$$

Now note since only one constraint is active this is trivially linearly independent and the point satisfies the necessary K-T conditions and is a min. for the function $f(x,y)$.

5. max $f(x,y) = xy$ subject to $g(x,y) = (x+y-2)^2 \le 0$

So,

$\mathcal{L} = xy - \lambda(x+y-2)^2$

$\frac{\partial \mathcal{L}}{\partial x} = y - 2\lambda(x+y-2) = 0$

$\frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda(x+y-2) = 0$

$\lambda \ge 0$ with $\lambda = 0$ if $(x+y-2)^2 < 0$

If $\lambda = 0$, then $y = x = 0$

and $4 < 0$ so there are no solutions if $\lambda = 0$.

If $\lambda \ne 0$, then $x = y$

and

$4(x-1)^2 = 0$

$x = 1 \Rightarrow y = 1$ however this yields a contradiction as $1 \ne 0$.

Thus, neither of the K-T conditions are satisfied. Also note

$\nabla g = \langle 2(x+y-2), 2(x+y-2) \rangle = \langle 0,0 \rangle$ at $(1,1)$ and the CQ conditions are not satisfied.

However, since $f(x,y)$ is monotonically increasing, we know that the solution has to be on the boundary where $(x+y-2)^2 = 0$. The solutions could also be negative, but because of the square these will not satisfy the constraint. Since this is true,

$f(x) = x(2-x)$ which is maximized when $x = y = 1$.

6. a) max $x^5 - y^3$ subject to: $x \le 1$, $x - y \le 0$

So,

$\mathcal{L} = x^5 - y^3 - \lambda_1(x-y) - \lambda_2(x-1)$

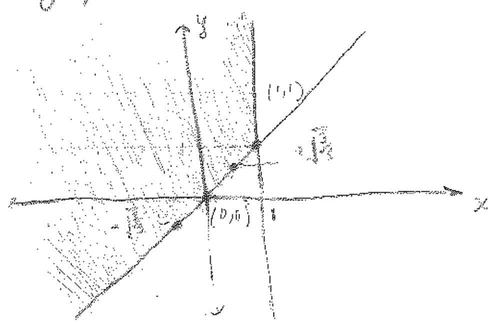
$\frac{\partial \mathcal{L}}{\partial x} = 5x^4 - \lambda_1 - \lambda_2 = 0$

$\frac{\partial \mathcal{L}}{\partial y} = -3y^2 + \lambda_1 = 0$

$\lambda_1 \ge 0$ with $\lambda_1 = 0$ if $x - y < 0$

$\lambda_2 \ge 0$ with $\lambda_2 = 0$ if $x < 1$

Since moving in the y direction only serves to decrease the value of $f(x,y)$ let us suppose the first constraint doesn't bind.
So, $\lambda_1 = 0$ which implies $y = 0$ and $x = 1$ which is not feasible.
Drawing the graph of the constraints:



Both constraints bind at the point $(1,1)$ which is obviously not a max as $f(1,1) = 0$. If we suppose neither constraint binds then $\lambda_1 = \lambda_2 = 0$ implies $x = y = 0$ and $f(0,0) = 0$ so this is not a maximum either. Let us suppose the second constraint doesn't bind $\lambda_2 = 0$ which implies that $x < 1$ and $x = y$. If this is true, then

$$5x^4 - 3x^2 = (5x^2 - 3)x^2 = 0$$

and either $x = 0 = y$ or $x = \pm\sqrt{\frac{3}{5}} = y$

$f(0,0) = 0$ and $f\left(-\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right) = -\left(\frac{3}{5}\right)^{5/2} + \left(\frac{3}{5}\right)^{3/2}$

and $f\left(\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right) = \left(\frac{3}{5}\right)^{5/2} - \left(\frac{3}{5}\right)^{3/2}$.

Since $f\left(-\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right) > f\left(\sqrt{\frac{3}{5}}, \sqrt{\frac{3}{5}}\right)$, the max occurs at $\left(-\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right)$
Also the the CQ conditions are satisfied as
$\vec{\nabla} g = \langle 1, -1 \rangle$ which is linearly independent.

b) First solve: $f(x) = x^5 - y^3$ subject to $x \le y$ where x is fixed and y varies
so max $x^5 - y^3$ subject to $-y \le -x$

$\mathcal{L} = x^5 - y^3 - \lambda(-y + x)$

$\frac{\partial \mathcal{L}}{\partial y} = -3y^2 + \lambda = 0$

$\lambda \ge 0$ with $\lambda = 0$ if $-y < -x$

If $\lambda = 0 \Rightarrow y = 0$ since $0 > -x$ this has no solutions.
So, $\lambda \ge 0$ and $y = x$. Thus,

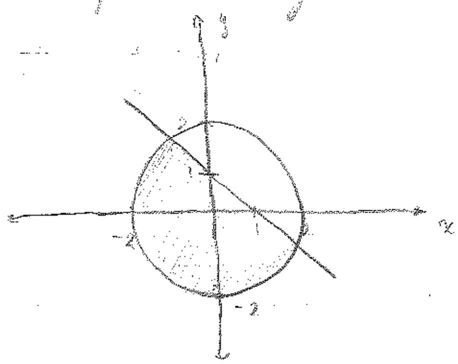
$f(x) = x^5 - x^3 \Rightarrow f'(x) = 5x^4 - 3x^2 = (5x^2-3)x^2 = 0$

either $x = 0$ or $x = \pm\sqrt{\frac{3}{5}}$

knowing the values from the previous problem we once again compare $f^*\left(-\sqrt{\frac{3}{5}}, -\sqrt{\frac{3}{5}}\right) = -\left(\frac{3}{5}\right)^{5/2} + \left(\frac{3}{5}\right)^{3/2}$ is the max.

2. max $xy + x + y$ subject to $x^2 + y^2 \leq 2$, $x + y \leq 1$

Note the feasible region is:

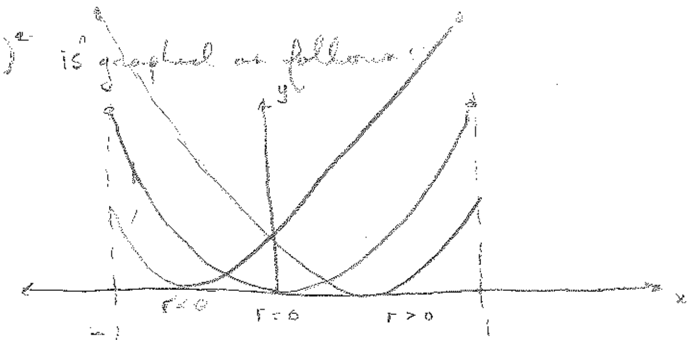


Note $f(x,y) = xy + x + y \leq xy + 1$ then at the maximum we wish to find the xy pair that maximizes this function where $x + y = 1$ it is clear enough to see that this happens at $x = y = \frac{1}{2}$ since

$$f'(x) = \frac{d}{dx}(x(1-x)) = 1 - 2x = 0 \Rightarrow x = \tfrac{1}{2}.$$

Noticing this we circumvent the KT conditions which would be rather cumbersome in this case.

2. Note $(x-r)^2$ is graphed as follows:



So if $r > 0$ then the value function is $(1+r)^2$

if $r < 0$ the value function is $(-1+r)^2$

if $r = 0$ the value function is 1

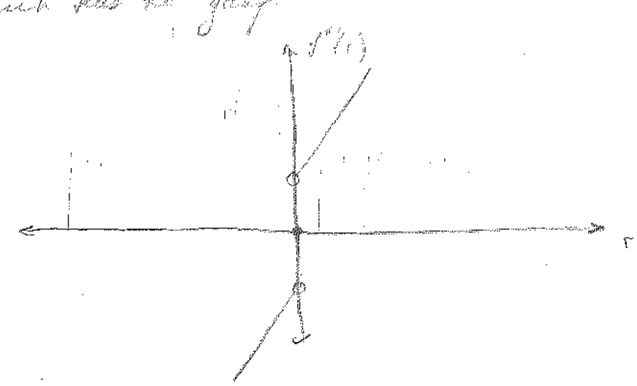
Then

$$f^*(r) = \begin{cases} (1+|r|)^2 & \text{if } |r| > 0 \\ 1 & \text{if } r = 0 \end{cases}$$

Note that

$$f^{*\prime}(r) = \begin{cases} 2\left(\frac{r}{|r|} + r\right)\frac{r}{-} & \text{if } |r| > 0 \\ 0 & \text{if } r = 0 \end{cases}$$

which has the graph



It is clear that the derivative is discontinuous at $r = 0$ and thus the function is not differentiable at $r = 0$.

2. a) max xy subject to $x + 2y \leq 2$, $x \geq 0$, $y \geq 0$

It is clear that this function will not be maximized at $x = 0, y = 0$. Thus, this reduces to a normal K-T problem, but we will ignore $(x, y) \in \mathbb{R}^-$.

Thus,

$$\mathcal{L} = xy - \lambda(x + 2y - 2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = y - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = x - 2\lambda = 0$$

$\lambda \geq 0$ or $\lambda = 0$ if $x + 2y < 2$

If $\lambda = 0 \Rightarrow y = x = 0$, $f(0,0) = 0$. This is clearly not the max.

If $\lambda > 0 \Rightarrow x = 2y \Rightarrow x = 1, y = \frac{1}{2}, \lambda = \frac{1}{2}$.

Hence $f(1, \frac{1}{2}) = \frac{1}{2}$. It is easy to see this is an optimum as $\nabla g = \langle 1, 2 \rangle$ which is linearly independent.

b) max $x^\alpha y^\beta$ subject to $x+2y \le 2$, $x>0$, $y>0$ where $\alpha>0$, $\beta>0$, and $\alpha+\beta \le 1$.

$$\mathcal{L} = x^\alpha y^\beta - \lambda(x+2y-2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \alpha x^{\alpha-1} y^\beta - \lambda = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = \beta x^\alpha y^{\beta-1} - 2\lambda = 0$$

$\lambda \ge 0$ or $\lambda = 0$ if $x+2y<2$

if $\lambda = 0 \Rightarrow \alpha \frac{f(x,y)}{x} = 0$ and $\beta \frac{f(x,y)}{y} = 0$

which is only satisfied if $x=0$ or $y=0$ which violates our second constraint.

So, $\lambda \ge 0$ and

$$\beta x^\alpha y^{\beta-1} = 2\alpha x^{\alpha-1} y^\beta$$

$$\Rightarrow \beta \frac{x}{y} = 2\alpha$$

$$\Rightarrow \beta x = 2\alpha y$$

Thus, $x + \frac{\beta}{\alpha} x = 2$

$$x^* = \frac{2\alpha}{(\alpha+\beta)} > 0,$$

and

$$y^* = \frac{\beta}{(\alpha+\beta)} > 0.$$

Once again, this is an optimal solution as $\vec{\nabla} g = \langle 1, 2 \rangle$ is lin. indep.

3. max $f(x,y) = cx+y$ subject to $g(x,y) = x^2+3y^2 \le 2$, $x \ge 0$, $y \ge 0$.

$\mathcal{L} = cx+y-\lambda(x^2+3y^2-2)$ where $c>0$

$\frac{\partial \mathcal{L}}{\partial x} = c - 2\lambda x \le 0$ if $x^* > 0$ $(=)$

$\frac{\partial \mathcal{L}}{\partial y} = 1 - 6\lambda y \le 0$ if $y^* > 0$ $(=)$

$\lambda \ge 0$ or $\lambda = 0$ if $x^2+3y^2<2$.

Since $f(x,y)$ is linear the optimum will be somewhere on the boundary so $\lambda \ge 0$ and $x^2+3y^2=2$. Furthermore the max will lie in the region $x \ge 0$, $y \ge 0$. Let us suppose $x>0$, $y>0$ so that

$$1 - \frac{3c}{x} y = 0 \Rightarrow x = 3cy$$

$$\Rightarrow 3(3c^2+1)y^2 = 2$$

$$\Rightarrow y^* = +\sqrt{\frac{2}{9c^2+3}}, \quad x^* = +\sqrt{\frac{2\cdot 9c^2}{9c^2+3}}.$$

As the function is increasing in both the x & y directions we can say that for $c > 0$ the solution is:

$$(x^*, y^*) = \left(+\sqrt{\frac{2 \cdot 9c^2}{9c^2+3}} , +\sqrt{\frac{2}{9c^2+3}} \right)$$

and $$f(x^*, y^*) = c\sqrt{\frac{18c^2}{9c^2+3}} + \sqrt{\frac{2}{9c^2+3}}$$

if $c = 0$

$$\mathcal{L} = y - \lambda(x^2 + 3y^2 - 2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = -2\lambda x \le 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = 1 - 6\lambda y = 0 \qquad \text{since } y^* > 0$$

$$\lambda \ge 0 \quad \text{or} \quad \lambda = 0 \text{ if } x^2 + 3y^2 < 2$$

since x doesn't matter set $x = 0$ so that

$$y = +\sqrt{\tfrac{2}{3}} \quad \text{and} \quad \lambda = +\tfrac{1}{6}\sqrt{\tfrac{3}{2}} > 0$$

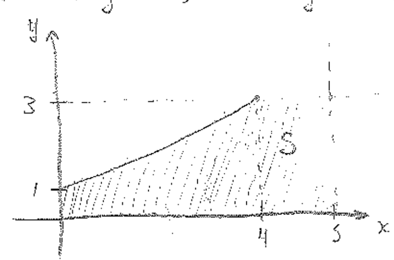
more generally $0 \le x \le \sqrt{2}$ since

$y = +\sqrt{\frac{2 - x^2}{3}}$ for any x in that range since it is more or less a constant. However this is clearly maximized at $x = 0$ where $y = +\sqrt{\frac{2}{3}}$.

If $c < 0$ then the max is trivially $x = 0$, $y = +\sqrt{\frac{2}{3}}$ since the function is decreasing across all positive x.

5. The feasible region is given by:

$$x \le 5,\ y \le 3,\ -x + 2y \le 2,\ x \ge 0,\ y \ge 0.$$



$y \le 1 + \frac{1}{2}x$

There are no stationary pts in this region.

Solve:

$$\mathcal{L} = \frac{3}{4}x + \frac{2}{3}y - \frac{1}{2}(x+y)^2 - \lambda_1(-x + 2y - 2) - \lambda_2(x - 5) - \lambda_3(y - 3)$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{3}{4} - x - y + \lambda_1 - \lambda_2 \quad (= 0 \text{ if } x^* > 0) \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{2}{3} - x - y - 2\lambda_1 - \lambda_3 \quad (= 0 \text{ if } y^* > 0) \qquad (2)$$

$$\lambda_1 \ge 0 \quad \text{or} \quad \lambda_1 = 0 \text{ if } -x^* + 2y^* < 2 \qquad (3)$$

$$\lambda_2 \geq 0 \text{ or } \lambda_2 = 0 \text{ if } x^* < 5 \quad (4)$$

$$\lambda_3 \geq 0 \text{ or } \lambda_3 = 0 \text{ if } y^* < 3 \,. \quad (5)$$

Let us suppose none of the constraints bind so that

$$\frac{3}{4} - x - y \leq 0$$

$$\frac{2}{3} - x - y \leq 0$$

Since the function is decreasing more rapidly in the y direction let us set $y = 0$ and go entirely in the x direction. Thus:

$$\frac{3}{4} - x = 0 \Rightarrow x = 3/4$$

$$\frac{2}{3} - x \leq 0$$ and the K-T conditions hold

Thus $(x^*, y^*) = (3/4, 0)$.