6.3

1e. $\ddot{x} + \dot{x} - 6x = 8$

First lets solve the general homogeneous equation

$r^2 + r - 6 = 0$

$(r-2)(r+3) = 0$

$r = 2, -3$

general form of homogeneous equation

$Ae^{2t} + Be^{-3t}$

Now let's deal with 8.

$u(t) = \frac{-8}{6} = \frac{-4}{3}$ is a solution

$\boxed{x = \frac{-4}{3} + Ae^{2t} + Be^{-3t}}$

Sangita Vyas

8.1

i. $x(t) = (e^2-1)t$

$\dot{x}(t) = e^2 - 1$

$J(x) = \int_0^1 [(e^2-1)^2 t^2 + (e^2-1)^2] dt = \frac{1}{3}(e^2-1)^2 t^3 + (e^2-1)^2 t \Big|_0^1 = \frac{1}{3}(e^2-1)^2 + (e^2-1)^2 - 0$

$= \boxed{\frac{4}{3}(e^2-1)^2 = J(x)}$

ii. $x(t) = e^{1+t} - e^{1-t}$

$\dot{x}(t) = e^{1+t} + e^{1-t}$

$J(x) = \int_0^1 [(e^{1+t} - e^{1-t})^2 + (e^{1+t} + e^{1-t})^2] dt = \int_0^1 \left(e^{2(1+t)} - \cancel{2e^2} + e^{2(1-t)} + e^{2(1+t)} + \cancel{2e^2} + e^{2(1-t)}\right) dt$

$= \frac{e^{2(1+t)}}{2} - \frac{e^{2(1-t)}}{2} + \frac{e^{2(1+t)}}{2} - \frac{e^{2(1-t)}}{2} \Big|_0^1$

$= \frac{e^4}{2} - \frac{1}{2} + \frac{e^4}{2} - \frac{1}{2} - \left(\frac{e^2}{2} - \frac{e^2}{2} + \frac{e^2}{2} - \frac{e^2}{2}\right) = \boxed{e^4 - 1 = J(x)}$

$e^4 - 1 < \frac{4}{3}(e^2-1)^2 \Rightarrow$ $\boxed{\text{function (ii) gives } J(x) \text{ lower value}}$

8.2

1. $F(t, x, \dot{x}) = 4xt - \dot{x}^2$

$\frac{\partial F}{\partial x} = 4t \quad \frac{\partial F}{\partial \dot{x}} = -2\dot{x} \quad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2\ddot{x}$

$4t + 2\ddot{x} = 0$

$\ddot{x} = -2t$

$\dot{x} = -t^2 + A$

$x = -\frac{1}{3}t^3 + At + B$

$x(0) = B = 2$

$x(1) = -\frac{1}{3} + A + 2 = \frac{2}{3}$

$A = -1$

$\boxed{x(t) = -\frac{1}{3}t^3 - t + 2}$

$F(t, x, \dot{x})$ is concave so this is the solution

2. $\min \int_0^1 (t\dot{x} + \dot{x}^2) dt$

$F(t, x, \dot{x}) = (t\dot{x} + \dot{x}^2)$

$\frac{\partial F}{\partial x} = 0 \quad \frac{\partial F}{\partial \dot{x}} = +t + 2\dot{x} \quad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = +1 + (2)(\ddot{x}) = +1 + 2\ddot{x}$

$0 - (1 + 2\ddot{x}) = -1 - 2\ddot{x} = 0$

$\ddot{x} = -\frac{1}{2}$

$\dot{x} = -\frac{1}{2}t + A$

$x = -\frac{1}{4}t^2 + At + B$

$x(0) = B = 1$

$x(1) = -\frac{1}{4} + A + 1 = 0$

$A = -\frac{3}{4}$

$\boxed{x = -\frac{1}{4}t^2 - \frac{3}{4}t + 1}$ f convex so this is solution

a. $F(t,x,\dot{x}) = x^2 + \dot{x}^2 + 2xe^t$

$\frac{\partial F}{\partial x} = 2x + 2e^t \qquad \frac{\partial F}{\partial \dot{x}} = 2\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2\ddot{x}$

Euler equation: $2x + 2e^t - 2\ddot{x} = 0$

$x + e^t - \ddot{x} = 0$

$\boxed{\ddot{x} - x = e^t}$

b. $F(t,x,\dot{x}) = -e^{\dot{x}-ax}$

$\frac{\partial F}{\partial x} = -e^{\dot{x}-ax}(-a) \qquad \frac{\partial F}{\partial \dot{x}} = -e^{\dot{x}-ax} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = ae^{\dot{x}-ax}(\dot{x}) + (-e^{\dot{x}-ax})\ddot{x}$

$ae^{\dot{x}-ax} - \dot{x}ae^{\dot{x}-ax} + \ddot{x}e^{\dot{x}-ax} = 0$

$a - \dot{x}a + \ddot{x} = 0$

$\boxed{a = \dot{x}a - \ddot{x}}$

c. $F(t,x,\dot{x}) = [(x-\dot{x})^2 + x^2]e^{-at} = e^{-at}(x-\dot{x})^2 + e^{-at}x^2$

$\frac{\partial F}{\partial x} = 2e^{-at}(x-\dot{x}) + 2e^{-at}x \quad \frac{\partial F}{\partial \dot{x}} = -2e^{-at}(x-\dot{x}) \quad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2ae^{-at}(x-\dot{x}) - 2e^{-at}\dot{x} + 2e^{-at}\ddot{x}$

$2e^{-at}(x-\dot{x}) + 2e^{-at}x - 2ae^{-at}(x-\dot{x}) + 2e^{-at}\dot{x} - 2e^{-at}\ddot{x} = 0$

$(x-\cancel{\dot{x}}) + x - a(x-\dot{x}) + \cancel{\dot{x}} - \ddot{x} = 0$

$x + x - ax + a\dot{x} - \ddot{x} = 0$

$\boxed{x(2-a) + a\dot{x} - \ddot{x} = 0}$

d. $F(t,x,\dot{x}) = 2tx + 3x\dot{x} + t\dot{x}^2$

$\frac{\partial F}{\partial x} = 2t + 3\dot{x} \qquad \frac{\partial F}{\partial \dot{x}} = 3x + 2t\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2\dot{x} + (3)\dot{x} + 2t\ddot{x}$

$2t + 3\dot{x} - 5\dot{x} - 2t\ddot{x} = 0$

$2t - 2\dot{x} - 2t\ddot{x} = 0$

$t\ddot{x} + \dot{x} = t$

$\boxed{\ddot{x} + \frac{1}{t}\dot{x} = 1}$

4. $F(t,x,\dot{x}) = x^2 + 2tx\dot{x} + \dot{x}^2$

$\frac{\partial F}{\partial x} = 2x + 2t\dot{x} \qquad \frac{\partial F}{\partial \dot{x}} = 2tx + 2\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2x + 2t\dot{x} + 2\ddot{x}$

$\cancel{2x} + \cancel{2t\dot{x}} - \cancel{2x} - \cancel{2t\dot{x}} - 2\ddot{x} = 0$

$-2\ddot{x} = 0$

$\ddot{x} = 0$

$\dot{x} = A$

$x = At + B$

$x(0) = B = 1$

$x(1) = A + 1 = 2 \Rightarrow A = 1$

$\boxed{x = t + 1}$ $\qquad$ F convex so this is min

5. $F(t,x,\dot{x}) = x^2 + tx + tx\dot{x} + \dot{x}^2$

$\frac{\partial F}{\partial x} = 2x + t + t\dot{x}$ $\quad \frac{\partial F}{\partial \dot{x}} = tx + 2\dot{x}$ $\quad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = x + t\dot{x} + 2\ddot{x}$

$2x + t + \cancel{t\dot{x}} - x - \cancel{t\dot{x}} - 2\ddot{x} = 0$

$x - 2\ddot{x} = -t$

$2\ddot{x} - x = t$

$\ddot{x} - \frac{1}{2}x = \frac{1}{2}t$

$r^2 - \frac{1}{2} = 0$

$\left(r - \frac{1}{\sqrt{2}}\right)\left(r + \frac{1}{\sqrt{2}}\right) = 0$

$x(t) = Ae^{\frac{1}{\sqrt{2}}t} + Be^{-\frac{1}{\sqrt{2}}t} + \frac{1}{2}\left(\frac{-2}{1}\right)t + 0$

$x(0) = A + B = 0$ $\qquad x(1) = Ae^{\frac{1}{\sqrt{2}}} + Be^{-\frac{1}{\sqrt{2}}} - 1 = 1$

$A = -B$ $\qquad -Be^{\frac{1}{\sqrt{2}}} + Be^{-\frac{1}{\sqrt{2}}} = 2$

$-B\left(e^{\frac{1}{\sqrt{2}}} + e^{-\frac{1}{\sqrt{2}}}\right) = 2$

$-B = \frac{2}{e^{\frac{1}{\sqrt{2}}} + e^{-\frac{1}{\sqrt{2}}}}$

$$\boxed{x(t) = \frac{2}{e^{\frac{1}{\sqrt{2}}} + e^{-\frac{1}{\sqrt{2}}}}\, e^{\frac{1}{\sqrt{2}}t} - \frac{2}{e^{\frac{1}{\sqrt{2}}} + e^{-\frac{1}{\sqrt{2}}}}\, e^{-\frac{1}{\sqrt{2}}t} - t}$$

F convex so this is min

6. $F(t,x,\dot{x}) = \sqrt{1+\dot{x}^2}$

$\frac{\partial F}{\partial x} = 0$ $\quad \frac{\partial F}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}$ $\quad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = \frac{\sqrt{1+\dot{x}^2} - \dot{x}^2(1+\dot{x}^2)^{-1/2}}{1+\dot{x}^2}\,\ddot{x}$

$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) = 0$

$\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = C$

$\dot{x} = C_1$

$x = C_1 t + C_2$

$x(t_0) = x_0 = C_1 t_0 + C_2$ $\qquad x(t_1) = x_1 = C_1 t_1 + C_2$

$C_2 = x_0 - C_1 t_0$ $\qquad x_1 = C_1 t_1 + x_0 - C_1 t_0$

$x_1 = C_1(t_1 - t_0) + x_0$

F convex so this is min

$\frac{x_1 - x_0}{t_1 - t_0} = C_1$

$C_2 = x_0 - t_0\left(\frac{x_1 - x_0}{t_1 - t_0}\right) = \frac{x_0(t_1 - t_0) - t_0(x_1 - x_0)}{t_1 - t_0}$

$= \frac{x_0 t_1 - \cancel{x_0 t_0} - t_0 x_1 + \cancel{t_0 x_0}}{t_1 - t_0}$

$= \frac{x_0 t_1 - t_0 x_1}{t_1 - t_0}$

$$\boxed{x(t) = \left(\frac{x_1 - x_0}{t_1 - t_0}\right)t + \frac{x_0 t_1 - t_0 x_1}{t_1 - t_0}}$$

this is linear - the shortest line connecting two points is the straight line

7. $F(x) = x^2 + tx\dot{x} + t^2\dot{x}^2 \qquad x(1) = 0 \quad x(2) = 1$

$\frac{\partial F}{\partial x} = 2x + t\dot{x} \qquad \frac{\partial F}{\partial \dot{x}} = tx + 2t^2\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = x + 4t\dot{x} + t\dot{x} + 2t^2\ddot{x}$

$2x + t\dot{x} - x - 5t\dot{x} - 2t^2\ddot{x} = 0$

$x - 4t\dot{x} - 2t^2\ddot{x} = 0$

$-2\left(t^2\ddot{x} + 2t\dot{x} - \frac{1}{2}x\right) = 0$

$x = t^r \qquad \dot{x} = rt^{r-1} \qquad \ddot{x} = r(r-1)t^{r-2}$

$t^2 r(r-1)t^{r-2} + 2trt^{r-1} - \frac{1}{2}t^r = 0$

$t^r\left(r^2 - r + 2r - \frac{1}{2}\right) = 0$

$r^2 + r - \frac{1}{2} = 0$

$r = \frac{-1 \pm \sqrt{1+2}}{2} = \frac{-1+\sqrt{3}}{2}, \frac{-1-\sqrt{3}}{2}$

$x = At^{\frac{-1+\sqrt{3}}{2}} + Bt^{\frac{-1-\sqrt{3}}{2}}$

$x(1) = A + B = 0 \Rightarrow A = -B,\ B = -A$

$x(2) = A\left(2^{\frac{-1+\sqrt{3}}{2}}\right) - A\left(2^{\frac{-1-\sqrt{3}}{2}}\right) = 1$

$A = \frac{1}{2^{\frac{-1+\sqrt{3}}{2}} - 2^{\frac{-1-\sqrt{3}}{2}}}$

$$\boxed{x(t) = \frac{1}{2^{\frac{-1+\sqrt{3}}{2}} - 2^{\frac{-1-\sqrt{3}}{2}}}\, t^{\frac{-1+\sqrt{3}}{2}} - \frac{1}{2^{\frac{-1+\sqrt{3}}{2}} - 2^{\frac{-1-\sqrt{3}}{2}}}\, t^{\frac{-1-\sqrt{3}}{2}}}$$

F is convex so this is the solution

8. $F(x) = [N(\dot{x}(t)) + \dot{x}(t)f(x(t))]e^{-rt} = [N(\dot{x}) + \dot{x}f(x)]e^{-rt}$

$\frac{\partial F}{\partial x} = \dot{x}f'(x)e^{-rt} \qquad \frac{\partial F}{\partial \dot{x}} = N'(\dot{x})e^{-rt} + f(x)e^{-rt} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = \frac{d}{dt}N'(\dot{x})e^{-rt} - rN'(\dot{x})e^{-rt} + f'(x)\dot{x}e^{-rt}$
$- rf(x)e^{-rt}$

$\cancel{\dot{x}f'(x)e^{-rt}} - \frac{d}{dt}N'(\dot{x})e^{-rt} + rN'(\dot{x})e^{-rt} - \cancel{f'(x)\dot{x}e^{-rt}} + rf(x)e^{-rt} = 0$

$-\frac{d}{dt}N'(\dot{x}) + rN'(\dot{x}) + rf(x) = 0$

$$\boxed{r(N'(\dot{x}) + f(x)) = \frac{d}{dt}N'(\dot{x})}$$