8.3

1. max $\int_0^1 (x^2 + \dot{x}^2)dt \quad x(0) = 0 \quad x(1) = 0$

Say $x(t) = a(t - t^2)$

$[x(t)]^2 = a^2(t - t^2)^2$

$\dot{x}(t) = a - 2at = a(1 - 2t)$

$[\dot{x}(t)]^2 = a^2(1 - 2t)^2$

$$\int_0^1 [a^2(t - t^2)^2 + a^2(1 - 2t)^2]dt = \int_0^1 [a^2((t^2 - 2t^3 + t^4) + (1 - 4t + 4t^2))]dt = \int_0^1 a^2(t^4 - 2t^3 + 5t^2 - 4t + 1)dt$$

$$= a^2\left(\frac{1}{5}t^5 - \frac{1}{2}t^4 + \frac{5}{3}t^3 - 2t^2 + t + c\right)\Big|_0^1 = a^2\left(\frac{6}{30} - \frac{15}{30} + \frac{50}{30} - \frac{60}{30} + \frac{30}{30} + c\right) - Ca^2 = \frac{11}{30}a^2$$

Is $x(t) = a(t - t^2)$ admissable?

$x(0) = 0$ ✓

$x(1) = 0$ ✓

Does it satisfy E-L equation?

$F(t, x, \dot{x}) = x^2 + \dot{x}^2$

$\frac{\partial F}{\partial x} = 2x \qquad \frac{\partial F}{\partial \dot{x}} = 2\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2\ddot{x}$

$2x - 2\ddot{x} = 0$

$\ddot{x} - x = 0$

$(\ddot{x}(t)) = -2a$

$-2a - a(t - t^2) = -a(2 + t - t^2) \neq 0$

$x(t) = a(t - t^2)$ is admissible because it satisfies boundary conditions but it is not maximum because it does not satisfy E-L. Moreover, as $a \to \infty$, $\frac{11}{30}a^2 \to \infty$, so there is no function that can maximize this integral.

2a. max $\int_0^T U(\bar{c} - \dot{x}e^{rt})dt \quad x(0) = x_0 \quad x(T) = 0$

$F(t, x, \dot{x}) = U(\bar{c} - \dot{x}e^{rt})$

$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial \dot{x}} = -e^{rt}U'(\bar{c} - \dot{x}e^{rt})$

$-\frac{d}{dt}\left(-e^{rt}U'(\bar{c} - \dot{x}e^{rt})\right) = 0$

$\boxed{e^{rt}U'(\bar{c} - \dot{x}e^{rt}) = K}$ K is constant

b. $U(c) = -\frac{e^{-vc}}{v} \qquad U'(c) = e^{-vc}$

$e^{rt}e^{-v(\bar{c} - \dot{x}e^{rt})} = K$

b. continued

$$e^{-v(\bar{c}-\dot{x}e^{rt})} = Ke^{-rt}$$

$$\ln e^{-v(\bar{c}-\dot{x}e^{rt})} = \ln K e^{-rt}$$

$$-v(\bar{c}-\dot{x}e^{rt}) = \ln K - rt$$

$$\dot{x} v e^{rt} = \ln K - rt + v\bar{c}$$

$$\dot{x} = \frac{\ln K - rt + v\bar{c}}{ve^{rt}}$$

$$\dot{x} = e^{-rt}\Big(\underbrace{\frac{\ln K}{v} + \bar{c}}_{=H} - \frac{rt}{v}\Big)$$

$$\dot{x} = e^{-rt}\left(H - \frac{rt}{v}\right)$$

$$x(t) = \int e^{-rt}\left(H - \frac{rt}{v}\right)dt \qquad u = \left(H - \frac{rt}{v}\right) \qquad dv = e^{-rt}$$

$$du = -\frac{r}{v}dt \qquad v = \frac{-e^{-rt}}{r}$$

$$= \frac{-e^{-rt}}{r}\left(H - \frac{rt}{v}\right) - \int \frac{-1}{r} e^{-rt}\left(\frac{-r}{v}\right)dt = \left(\frac{t}{v} - \frac{H}{r}\right)e^{-rt} + \frac{1}{rv}e^{-rt}$$

$$\boxed{x(t) = \frac{tr - Mv + 1}{rv}e^{-rt} + C \qquad M = \frac{\ln K}{v} + \bar{c} \quad C \text{ is found by } x(0) = x_0 \quad x(T) = 0}$$

This solution is optimal because $U(c)$ is concave

3. $\min \int_a^1 t\dot{x}^2 \, dt \quad x(a)=0 \quad x(1)=1$

$F = t\dot{x}^2$

$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial \dot{x}} = 2t\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2\dot{x} + 2t\ddot{x}$

$\frac{d}{dt}(2t\dot{x}) = 0$

$t\dot{x} = k$

$\dot{x} = kt^{-1}$

$x = k\ln t + C$

$x(1) = k\ln 1 + C = 1$

$C = 1$

$x(a) = k\ln a + 1 = 0$

$k = \frac{-1}{\ln a}$

$\boxed{x(t) = \frac{-\ln t}{\ln a} + 1}$ F is convex so this is solution for $a \in (0,1)$

When $a=0$, $x(0) \neq 0$. In fact $x(0)$ cannot be calulated. Thus $a \neq 0$ but it may be $(0,1)$.

4a. $F(t,y,\dot{y}) = \ln[y - \sigma\dot{y} - z l(t)]$

$\frac{\partial F}{\partial y} = \frac{1}{y - \sigma\dot{y} - z l(t)} \qquad \frac{\partial F}{\partial \dot{y}} = \frac{-\sigma}{y - \sigma\dot{y} - z l(t)}$

$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) = +\sigma(y - \sigma\dot{y} - z l(t))^{-2} z l'(t) - \sigma(y - \sigma\dot{y} - z l'(t))^{-2}\dot{y} + \sigma^2(y - \sigma\dot{y} - z l(t))^{-2}\ddot{y}$

$(y - \sigma\dot{y} - z l(t))^{-1} - \sigma(\dot{y} - \sigma\ddot{y} - z l'(t))(y - \sigma\dot{y} - z l(t))^{-2} = 0$

$\cancel{y - \sigma\dot{y}} - z l(t) - \sigma\dot{y} + \sigma^2\ddot{y} + z\sigma l'(t) = 0$

$\sigma^2\ddot{y} - 2\sigma\dot{y} + y = z l(t) - z\sigma l'(t)$

$\boxed{\ddot{y} - \frac{2}{\sigma}\dot{y} + \frac{y}{\sigma^2} = \frac{z l(t)}{\sigma^2} - \frac{z l'(t)}{\sigma}}$

b. $l(t) = l_0 e^{\alpha t} \qquad l'(t) = l_0 \alpha e^{\alpha t}$

$\ddot{y} - \frac{2}{\sigma}\dot{y} + \frac{1}{\sigma^2} y = \frac{z l_0 e^{\alpha t}}{\sigma^2} - \frac{z l_0 \alpha e^{\alpha t}}{\sigma} = z l_0 e^{\alpha t}\left(\frac{1}{\sigma^2} - \frac{\alpha}{\sigma}\right)$

First lets find general solution as if it were homogeneous

$$r^2 - \frac{2}{\sigma} r + \frac{1}{\sigma^2} = 0$$

$$(r - \tfrac{1}{\sigma})(r - \tfrac{1}{\sigma}) = 0$$

$$r = \frac{1}{\sigma}$$

$$Ae^{\frac{1}{\sigma}t} + Bte^{\frac{1}{\sigma}t}$$

Now lets find a specific solution, will be of the form

$$y(t) = Ce^{\alpha t}$$

$$\dot{y}(t) = C\alpha e^{\alpha t}$$

$$\ddot{y}(t) = C\alpha^2 e^{\alpha t}$$

Now plug into E-L

$$C\alpha^2 e^{\alpha t} - \frac{2}{\sigma} C\alpha e^{\alpha t} + \frac{1}{\sigma^2} Ce^{\alpha t} = C\cancel{e^{\alpha t}}\left(\alpha^2 - \frac{2\alpha}{\sigma} + \frac{1}{\sigma^2}\right) = 2l_0 \cancel{e^{\alpha t}}\left(\frac{1}{\sigma^2} - \frac{\alpha}{\sigma}\right)$$

$$C = \frac{2l_0\left(\frac{1}{\sigma^2} - \frac{\alpha}{\sigma}\right)}{\alpha^2 - \frac{2\alpha}{\sigma} + \frac{1}{\sigma^2}} = \frac{\frac{2l_0}{\cancel{\sigma^2}}(1 - \alpha\sigma)}{\frac{1}{\cancel{\sigma^2}}(\alpha^2\sigma^2 - 2\alpha\sigma + 1)} = \frac{2l_0(1 - \cancel{\alpha\sigma})}{(1-\alpha\sigma)(\cancel{1-\alpha\sigma})}$$

$$= \frac{2l_0}{1-\alpha\sigma}$$

$$\boxed{y(t) = \frac{2l_0}{1-\alpha\sigma} e^{\alpha t} + Ae^{\frac{1}{\sigma}t} + Bte^{\frac{1}{\sigma}t}}$$

1

$$F(t, k, \dot{k}) = e^{-t/4} \ln(2k - \dot{k})$$

$$\frac{\partial F}{\partial k} = \frac{2e^{-t/4}}{2k - \dot{k}} \qquad \frac{\partial F}{\partial \dot{k}} = \frac{-e^{-t/4}}{2k - \dot{k}} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{k}}\right) = \frac{e^{-t/4}}{4(2k-\dot{k})} + \frac{2e^{-t/4}\dot{k}}{(2k-\dot{k})^2} - \frac{e^{-t/4}\ddot{k}}{(2k-\dot{k})^2}$$

$$\frac{2\cancel{e^{-t/4}}}{2k-\dot{k}} - \frac{\cancel{e^{-t/4}}}{4(2k-\dot{k})} - \frac{\dot{k}\,2\cancel{e^{-t/4}}}{(2k-\dot{k})^2} + \frac{\ddot{k}\cancel{e^{-t/4}}}{(2k-\dot{k})^2} = 0$$

$$\frac{2}{2k-\dot{k}} - \frac{1}{4(2k-\dot{k})} - \frac{(2\dot{k} - \ddot{k})}{(2k-\dot{k})^2} = 0$$

$$\frac{2(4(2k-\dot{k})) - (2k-\dot{k}) - 4(2\dot{k} - \ddot{k})}{4\cancel{(2k-\dot{k})^2}} = 0$$

$$16k - 8\dot{k} - 2k + \dot{k} - 8\dot{k} + 4\ddot{k} = 0$$

$4\ddot{k} - 15\dot{k} + 14k = 0 \Rightarrow$ characteristic polynomial : $4r^2 - 15r + 14 = 0 \Rightarrow r = \frac{15 \pm \sqrt{225 - 224}}{8}$

$$r = 2, \frac{7}{4}$$

1. continued

$Ae^{2t} + Be^{7/4t} = K(t)$

$K(0) = A + B = K_0 \quad K(T) = Ae^{2T} + (K_0 - A)e^{7/4T} = K_T$

$B = K_0 - A$

$$A = \frac{K_T - K_0 e^{7/4T}}{e^{2T} - e^{7/4T}}$$

$$B = \frac{K_0 e^{2T} - \cancel{K_0 e^{7/4T}} - K_T + \cancel{K_0 e^{7/4T}}}{e^{2T} - e^{7/4T}}$$

$$\boxed{\frac{K_T - K_0 e^{7/4T}}{e^{2T} - e^{7/4T}} e^{2t} + \frac{K_0 e^{2T} - K_T}{e^{2T} - e^{7/4T}} e^{7/4t} = K(t)}$$

F concave so this is optimal

2a. $F(t, x, \dot{x}) = e^{-t/10}\left(\frac{1}{100}tx - \dot{x}^2\right)$

$\frac{\partial F}{\partial x} = \frac{1}{100}te^{-t/10} \qquad \frac{\partial F}{\partial \dot{x}} = -2e^{-t/10}\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = \frac{1}{5}e^{-t/10}\dot{x} - 2e^{-t/10}\ddot{x}$

$\frac{1}{100}t\cancel{e^{-t/10}} - \frac{1}{5}\cancel{e^{-t/10}}\dot{x} + 2\cancel{e^{-t/10}}\ddot{x} = 0$

$2\ddot{x} - \frac{1}{5}\dot{x} = \frac{-t}{100}$

$\ddot{x} - \frac{1}{10}\dot{x} = \frac{-t}{200}$

characteristic polynomial : $r^2 - \frac{1}{10}r = 0$

$r(r - \frac{1}{10}) = 0$

$r = 0, \frac{1}{10}$

$A + Be^{\frac{1}{10}t}$

Now for a specific solution

$x = Ct^2 + Dt$

$\dot{x} = 2Ct + D$

$\ddot{x} = 2C$

$2C - \frac{1}{10}(2Ct + D) = \frac{-t}{200}$

$2C - \frac{1}{5}Ct - \frac{1}{10}D = \frac{-t}{200}$

$x(0) = 0 \Rightarrow 2C - \frac{1}{10}D = 0$

$C = \frac{1}{20}D \Rightarrow 20C = D$

$x(T) = S \Rightarrow 2C - \frac{CT}{5} - 2C = \frac{-T}{200}$

$C = \frac{1}{40} \quad D = \frac{1}{2}$

2a. continued

$x(t) = A + Be^{\frac{1}{10}t} + \frac{1}{40}t^2 + \frac{1}{2}t$

$x(0) = A + B = 0 \qquad x(T) = A - Ae^{\frac{1}{10}T} + \frac{T^2}{40} + \frac{T}{2} = S$

$A = -B \qquad A - Ae^{\frac{1}{10}T} = S - \frac{T^2}{40} - \frac{T}{2}$

$$A = \frac{S - \frac{T^2}{40} - \frac{T}{2}}{1 - e^{\frac{1}{10}T}}$$

$$\boxed{x(t) = \frac{S - \frac{T^2}{40} - \frac{T}{2}}{1 - e^{\frac{1}{10}T}} - e^{\frac{1}{10}t}\left[\frac{S - \frac{T^2}{40} - \frac{T}{2}}{1 - e^{\frac{1}{10}T}}\right] + \frac{t^2}{40} + \frac{t}{2}}$$

F concave so this is optimal

2b. $T = 10,\ S = 20$

$$x(t) = \frac{20 - \frac{5}{2} - 5}{1 - e} - e^{\frac{t}{10}}\left[\frac{25}{2(1-e)}\right] + \frac{t^2}{40} + \frac{t}{2}$$

$$\boxed{x(t) = \frac{25}{2(1-e)}\left(1 - e^{\frac{t}{10}}\right) + \frac{t^2}{40} + \frac{t}{2}}$$

3. $F(t, K, \dot{K}) = u(f(K,t) - \dot{K} - \delta K, t)$

$$\frac{\partial F}{\partial K} = \underbrace{u'(f(K,t) - \dot{K} - \delta K, t)}_{u'_c}(f'(K,t) - \delta) \qquad \frac{\partial F}{\partial \dot{K}} = -\underbrace{u'(f(K,t) - \dot{K} - \delta K, t)}_{u'_c}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{K}}\right) = -u''_{ct} - u''_{cc}(f'(K,t) - \delta)\dot{K} + u''_{cc}\ddot{K} = -u''_{ct} - u''_{cc}\underbrace{\left[(f'(K,t) - \delta)\dot{K} - \ddot{K}\right]}_{\dot{c}}$$

$$u'_c(\underbrace{f'(K,t)}_{f'_K} - \delta) + u''_{ct} + u''_{cc}\dot{c} = 0$$

$$\boxed{\dot{c} = \frac{-u'_c(f'_K - \delta) - u''_{ct}}{u''_{cc}}}$$

$$\frac{\dot{c}}{c} = \frac{-(u'_c(f'_K - \delta) + u''_{ct})}{c\, u''_{cc}}$$

$$\boxed{\frac{\dot{c}}{c} = -\frac{1}{W}\left(\frac{u''_{ct}}{u'_c} + f'_K - \delta\right)}$$

1a. $F(t,p,\dot p) = pD(p,\dot p) - b(D(p,\dot p))$

$$\frac{\partial F}{\partial p} = D + pD_p' - b'(D)D_p' \qquad \frac{\partial F}{\partial \dot p} = pD_{\dot p}' - b'(D)D_{\dot p}'$$

$$\boxed{D + D_p'(p - b'(D)) - \frac{d}{dt}\left[D_{\dot p}'(p - b'D)\right] = 0}$$

b. $b(x) = \alpha x^2 + \beta x + \gamma \qquad x = D(p,\dot p) = Ap + B\dot p + C$

$D_p' = A$

$b'(x) = 2\alpha x + \beta = 2\alpha(Ap + B\dot p + C) + \beta$

$D_{\dot p}' = B$

$$\frac{d}{dt}\left(B\left(p - 2\alpha Ap - 2\alpha B\dot p - 2\alpha C - \beta\right)\right)$$
$$= B(\dot p - 2\alpha A\dot p - 2\alpha B\ddot p)$$

E-L: $Ap + B\dot p + C + A(p - 2\alpha Ap - 2\alpha B\dot p - 2\alpha C - \beta) - B(\dot p - 2A\alpha\dot p - 2B\alpha\ddot p) = 0$

$$Ap + \cancel{B\dot p} + C + Ap - 2\alpha A^2 p - \cancel{2\alpha AB\dot p} - 2\alpha AC - A\beta - \cancel{B\dot p} + \cancel{2AB\alpha\dot p} + 2B^2\alpha\ddot p = 0$$

$$2B^2\alpha\ddot p + p(2A - 2\alpha A^2) = 2\alpha AC + A\beta - C$$

$$\ddot p + p\left(\frac{A - \alpha A^2}{B^2\alpha}\right) = \frac{2\alpha AC + A\beta - C}{2B^2\alpha}$$

If its homogenous:

$$r^2 + \frac{A - \alpha A^2}{B^2\alpha} = 0$$

$$r = \pm\sqrt{\frac{A^2\alpha - A}{B^2\alpha}}$$ this is >0 since $A<0$, the root >0.

Specific solution, p a constant

$$p = \frac{2\alpha AC + A\beta - C}{2\cancel{B^2\alpha}}\left(\frac{\cancel{B^2\alpha}}{A - \alpha A^2}\right) = \frac{2\alpha AC + A\beta - C}{2(A - \alpha A^2)}$$

$$\boxed{p(t) = Ae^{t\sqrt{\frac{A^2\alpha - A}{B^2\alpha}}} + Be^{-t\sqrt{\frac{A^2\alpha - A}{B^2\alpha}}} + \frac{2\alpha AC + A\beta - C}{2(A - \alpha A^2)}}$$

A and B given by $P(0) = P_0$, $P(T) = P_T$