9.5

1. min $\int_0^1 (t\dot{x} + \dot{x}^2)\,dt \qquad x(0)=1$

$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial \dot{x}} = t + 2\dot{x} \qquad \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial \dot{x}}\right) = 1 + 2\ddot{x}$

$\frac{\partial}{\partial t}(t + 2\dot{x}) = 0$

$t + 2\dot{x} = C$

$\dot{x} = \frac{C-t}{2}$

$x = \frac{C}{2}t - \frac{1}{4}t^2 + D$

$x(0) = 1 = D$

a) $x(1)$ free

$x(t) = \frac{C}{2}t - \frac{1}{4}t^2 + 1$

$\left.\frac{\partial F}{\partial \dot{x}}\right|_{t=1} = 1 + C - 1 = C = 0$

$\boxed{x(t) = -\frac{1}{4}t^2 + 1}$ F combination of 2 convex functions in $\dot{x} \Rightarrow$ F convex and this solution is optimal.

b) $x(1) \geq 1$

$\left.\frac{\partial F}{\partial \dot{x}}\right|_{t=1} = 0$ if $x(1) > 1$

$C = 0$

$x(1) = -\frac{1}{4} + 1 < 1 \Rightarrow$ not possible

$\left.\frac{\partial F}{\partial \dot{x}}\right|_{t=1} < 0$ if $x(1) = 1$

$x(1) = \frac{C}{2} - \frac{1}{4} + \cancel{1} = \cancel{1}$

$\frac{C}{2} = \frac{1}{4} \Rightarrow C = \frac{1}{2}$

$\boxed{x(t) = \frac{1}{4}t - \frac{1}{4}t^2 + 1}$

2. max $\int_0^1 (10 - \dot{x}^2 - 2x\dot{x} - 5x^2)e^{-t}\,dt \qquad x(0)=0 \quad x(1)=1$

$\frac{\partial F}{\partial x} = (-2\dot{x} - 10x)e^{-t} \qquad \frac{\partial F}{\partial \dot{x}} = (-2\dot{x} - 2x)e^{-t} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -e^{-t}(-2\dot{x} - 2x) + e^{-t}(-2\ddot{x} - 2\dot{x})$

$(-2\dot{x} - 10x)\cancel{e^{-t}} + (-2\dot{x} - 2x)\cancel{e^{-t}} - \cancel{e^{-t}}(-2\ddot{x} - 2\dot{x}) = 0$

$-\dot{x} - 5x - \cancel{\dot{x}} - x + \ddot{x} + \cancel{\dot{x}} = 0$

$\ddot{x} - \dot{x} - 6x = 0 \Rightarrow (r-3)(r+2) = 0$

$Ae^{3t} + Be^{-2t} = x(t)$

$x(0) = A + B = 0 \Rightarrow A = -B \qquad x(1) = -Be^{3} + Be^{-2} = 1 \Rightarrow B = \frac{1}{e^{-2} - e^{3}}$

ontinued

$$\boxed{x(t) = \frac{e^{-2t} - e^{3t}}{e^{-2} - e^{3}}}$$

$F'' = \begin{matrix} -10 & -2 \\ -2 & -2 \end{matrix}$ $\left.\begin{matrix} D_1 = -10 \\ D_2 = 16 \end{matrix}\right\}$ F concave in $x, \dot{x}$ so solution is optimal

a) $x(1)$ free

$\left.\frac{\partial F}{\partial \dot{x}}\right|_{t=1} = 0$ $\quad x = -Be^{3t} + Be^{-2t}$

$\dot{x} = -3Be^{3t} - 2Be^{-2t}$

$\frac{\partial F}{\partial \dot{x}} = \left.\left[-2(-3Be^{3t} - 2Be^{-2t}) - 2(-Be^{3t} + Be^{-2t})\right]e^{-t}\right|_{t=1} = 0$

$\left[-\cancel{2}(-3Be^{3} - 2Be^{-2}) - \cancel{2}(-Be^{3} + Be^{-2})\right]\cancel{e^{-1}} = 0$

$B = -A = 0$

$$\boxed{x(t) = 0}$$

b) $x(1) \geq 2$

$\left.\frac{\partial F}{\partial \dot{x}}\right|_{t=1} = 0$ if $x(1) > 2$

$\Rightarrow$ not possible since $x(1) = 0$

$x(1) = 2$

$-Be^{3} + Be^{-2} = 2$

$B = \frac{2}{e^{-2} - e^{3}}$

$$\boxed{x(t) = \frac{2e^{-2t} - 2e^{3t}}{e^{-2} - e^{3}}}$$

.2

$\max \int_0^2 [e^t x(t) - u(t)^2]\, dt \quad \dot{x}(t) = -u(t) \quad u(t) \in (-\infty, \infty)$

$H(t, x, u, p) = e^t x - u^2 - pu$

$\frac{\partial H}{\partial u} = -2u - p$

$-2u^* - p = 0$

$u^*(t) = \frac{-p(t)}{2}$

H is concave function of u so $\frac{\partial H}{\partial u} = 0$ finds optimal solution

$\dot{p}(t) = \frac{-\partial H}{\partial x} = -e^t$

$p(t) = -e^t + C$

$x(2)$ free $\Rightarrow p(2) = 0 \Rightarrow -e^2 + C = 0 \Rightarrow C = e^2 \Rightarrow p(t) = -e^t + e^2 \Rightarrow u^*(t) = \frac{e^t}{2} - \frac{e^2}{2}$

...tinued

$x^*(t) = -\frac{1}{2}e^t + \frac{e^2}{2}t + D$

$x(0) = -\frac{1}{2} + D = 0 \Rightarrow D = \frac{1}{2}$

$$\boxed{x^*(t) = \frac{e^2}{2}t - \frac{e^t}{2} + \frac{1}{2} \qquad u^*(t) = \frac{e^t}{2} - \frac{e^2}{2}}$$

$H'' = \begin{matrix} 0 & 0 \\ 0 & -2 \end{matrix}$ H is concave in x, u so this solution is optimal

2. $\int_0^1 (1-u^2)\,dt \qquad \dot{x}(t) = x(t) + u(t) \qquad u \in (-\infty, \infty)$

$H = 1 - u^2 + px + pu$

$\frac{\partial H}{\partial u} = -2u + p \Rightarrow u^*(t) = \frac{p(t)}{2}$

H sum of concave functions in u so u^* is optimal

$\dot{p}(t) = -\frac{\partial H}{\partial x} = -p(t)$

$p(1) = 0 \Rightarrow p(t) = 0 \Rightarrow u^*(t) = 0$

$\dot{x}^*(t) = x^*(t)$

$x(0) = 1 \Rightarrow \boxed{x^*(t) = e^t, \; u^*(t) = 0}$

H sum of concave and affine functions of u

affine function of x

$\Rightarrow$ H concave in $x, u \Rightarrow$ solution is optimal

$\int_0^1 (x + u^2)\,dt \qquad \dot{x} = -u \qquad u \in (-\infty, \infty)$

$H = x + u^2 - pu$

$\frac{\partial H}{\partial u} = 2u - p \qquad u^*(t) = \frac{p(t)}{2}$ H convex in u, so u^* optimal

$\dot{p}(t) = -\frac{\partial H}{\partial x} = -1$

$p(t) = -t + C$

$x(1)$ free $\Rightarrow p(1) = 0 = C = 1$

$p(t) = -t + 1 \qquad u^*(t) = -\frac{1}{2}t + \frac{1}{2}$

$x^*(t) = +\frac{1}{4}t^2 - \frac{1}{2}t + D$

$x(0) = D = 0$

$$\boxed{x^*(t) = \frac{1}{4}t^2 - \frac{1}{2}t, \quad u^*(t) = -\frac{1}{2}t + \frac{1}{2}}$$

H is convex in x

sum of convex and affine in u

$\Rightarrow$ solution is optimal

$\max \int_0^{10} (1-4x-2u^2)\,dt \quad \dot{x}=u \quad u\in(-\infty,\infty)$

$H = 1-4x-2u^2+pu$

$\frac{\partial H}{\partial u} = -4u+p \quad u^*(t)=\frac{p(t)}{4}$

H is sum of concave and affine in $u \Rightarrow$ H concave in $u \Rightarrow u^*$ optimal

$\dot{p}(t) = -\frac{\partial H}{\partial x} = 4$

$p(t) = 4t + C$

$x(10)$ free $\Rightarrow p(10)=0 \Rightarrow C=-40$

$p(t)=4t-40 \quad u^*(t)=t-10$

$x^*(t) = \frac{1}{2}t^2 - 10t + D$

$x(0)=D=0$

$$\boxed{x^*(t)=\frac{1}{2}t^2-10t \qquad u^*(t)=t-10}$$

$H'' = \begin{matrix} -4 & 0 \\ 0 & 0 \end{matrix}$ H concave in $u, x \Rightarrow$ solution is optimal

$\int_0^T (x-u^2)\,dt \quad \dot{x}=x+u \quad u\in(-\infty,\infty)$

$H = x-u^2+px+pu$

$\frac{\partial H}{\partial u} = -2u+p \quad u^*(t)=\frac{p(t)}{2}$ H concave in u so u^* optimal

$\dot{p}(t) = -\frac{\partial H}{\partial x} = -1-p$

$\dot{p}+p=-1$

$r=-1 \Rightarrow Ae^{-t}-1=p(t)$

$x(T)$ free $\Rightarrow p(T)=Ae^{-T}-1=0 \Rightarrow A=\frac{1}{e^{-T}}=e^T$

$p(t)=e^Te^{-t}-1 = e^{T-t}-1$

$u^*(t) = \frac{e^{T-t}-1}{2}$

$\dot{x}^*(t) = x^*(t) + \frac{e^{T-t}-1}{2} \Rightarrow \dot{x}-x = \frac{1}{2}e^Te^{-t}-\frac{1}{2}$

general solution: Be^t

specific solution: $Ce^{-t}+D=x \Rightarrow \dot{x}=-Ce^{-t}$

$-Ce^{-t}-Ce^{-t}-D = \frac{1}{2}e^Te^{-t}-\frac{1}{2}$

$C=-\frac{1}{4}e^T \quad D=\frac{1}{2}$

$Be^t-\frac{1}{4}e^{T-t}+\frac{1}{2}=x(t) \quad x(0)=B-\frac{1}{4}e^T+\frac{1}{2}=0 \Rightarrow B=\frac{1}{4}e^T-\frac{1}{2}=\frac{e^T-2}{4}$

$$\boxed{x^*(t)=\frac{1}{4}e^{T+t}-\frac{1}{4}e^{T-t}-\frac{1}{2}e^t+\frac{1}{2} \qquad u^*(t)=\frac{e^{T-t}-1}{2}}$$

H concave in $x,u \Rightarrow$ solution is optimal

$\boxed{u^*(t) = 1}$

$\dot{x}^*(t) = 1 \Rightarrow x^*(t) = t + C$

$x(0) = 0 \Rightarrow C = 0$

$\boxed{x^*(t) = t}$

$V = \int_0^T t\,dt = \frac{1}{2}t^2\Big|_0^T = \boxed{\frac{1}{2}T^2 = V(T)}$

$H = x + pu$

$\dot{p} = -\frac{\partial H}{\partial x} = -1$

$p = -t + C \qquad x(T)$ free $\Rightarrow p(T) = 0$

$p(T) = -T + C = 0$

$C = T$

$p(t) = -t + T$

Want to maximize H so u should be as big as possible so $u^*(t) = 1$

$x(t) = t$

H is linear in $x, u \Rightarrow$ concave

This solution is optimal

$\max \int_0^1 (1 - x^2 - u^2)\,dt \quad \dot{x} = u \quad x(0) = 0 \quad x(1) \geq 1 \quad u \in \mathbb{R}$

regular control theory problem

$H = 1 - x^2 - u^2 + pu$

$\frac{\partial H}{\partial u} = -2u + p \qquad u^*(t) = \frac{p(t)}{2}$

H sum of concave in u & linear in $u \Rightarrow H$ concave in $u \Rightarrow u^*$ optimal

$\dot{p} = -\frac{\partial H}{\partial x} = 2x$

$\ddot{p} = 2\dot{x}$

$\dot{x} = u = \frac{p(t)}{2}$

$\ddot{p} = p \Rightarrow \ddot{p} - p = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow r = \pm 1$

$p = Ae^t + Be^{-t} \qquad \dot{x} = \frac{1}{2}(Ae^t + Be^{-t})$

$x = \frac{1}{2}(Ae^t - Be^{-t})$

$x(0) = \frac{1}{2}(A - B) = 0 \Rightarrow A = B$

$x(1) \geq 1 \Rightarrow p(1) \geq 0$ and in equality if $x(1) > 1$

$p(1) = 0 \Rightarrow A = 0 \Rightarrow x(1) = 0 \Rightarrow$ contradiction,

$x(1) = 1 = \frac{1}{2}(Ae - Ae^{-1}) \Rightarrow A = \frac{2}{e - e^{-1}}$

$$\boxed{\begin{aligned} x(t) &= \frac{1}{e - e^{-1}}\left(e^t - e^{-t}\right) \\ u(t) &= \frac{1}{e - e^{-1}}\left(e^t + e^{-t}\right) \end{aligned}}$$

H concave in x, u so this is optimal

6. max $\int_0^5 [10u - u^2 - 2] e^{-.1t}$ $\quad \dot{x} = -u$, $x(0) = 10$, $x(5) \geq 0$, $u \geq 0$

$H = (10u - u^2 - 2)e^{-.1t} - pu$

$\dot{p} = \frac{-\partial H}{\partial x} = -(0)$

$p = k$

Suppose $u^*(t) \geq 0 \quad \forall t\ [0,5]$

$\frac{\partial H}{\partial u} = (10 - 2u^*)e^{-.1t} - p = 0$ $\quad$ H concave in u so u^* optimal

$(10 - 2u^*)e^{-.1t} = k$

Now transversality

$x(5) \geq 0 \Rightarrow p \geq 0$ and inequality if $x(5) > 0$

what if $p = 0$

$(10 - 2u^*)e^{-.1t} = 0$

$u^* = 5 > 0$ ✓

$\dot{x} = -5$

$x = -5t + C \Rightarrow x = -5t + 10$

$x(5) = -25 + 10 < 0 \Rightarrow$ contradiction

$p > 0$, $x(5) = 0$

$(10 - 2u^*)e^{-.1t} = k$

$u^* = \frac{-k}{2e^{-.1t}} + 5 = 5 - \frac{k}{2}e^{.1t}$

$\dot{x}^* = \frac{k}{2}e^{.1t} - 5$

$x = 5ke^{.1t} - 5t + C$

$x(0) = 5k + C = 10$

$C = 10 - 5k$

$x(5) = 5ke^{\frac{1}{2}} - 25 + 10 - 5k = 0$

$5ke^{\frac{1}{2}} - 5k = 15$

$k(5e^{\frac{1}{2}} - 5) = 15$

$k = \frac{15}{5e^{\frac{1}{2}} - 5} = \frac{3}{e^{\frac{1}{2}} - 1}$

H concave in x, u
(x not in function)
$\Rightarrow$ solution optimal

$$\boxed{x(t) = \frac{15}{e^{\frac{1}{2}} - 1} e^{.1t} - 5t + 10 - \frac{15}{e^{\frac{1}{2}} - 1} \qquad u(t) = 5 - \frac{3}{2(e^{\frac{1}{2}} - 1)} e^{.1t}}$$

1. max $\int_0^1 \left(2xe^{-t} - 2x\dot{x} - \dot{x}^2\right) dt \qquad x(0) = 0 \qquad x(1) = 1$

CoV

$\frac{\partial F}{\partial x} = 2e^{-t} - 2\dot{x} \qquad \frac{\partial F}{\partial \dot{x}} = -2x - 2\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2\dot{x} - 2\ddot{x}$

$2e^{-t} - 2\dot{x} - (-2\dot{x} - 2\ddot{x}) = 0$

$\cancel{2}e^{-t} + \cancel{2}\ddot{x} = 0$

$\ddot{x} = -e^{-t}$

$\dot{x} = e^{-t} + C$

$x = -e^{-t} + Ct + D$

$x(0) = -1 + D \Rightarrow D = 1$

$x(1) = -e^{-1} + C + \cancel{1} = \cancel{1} \Rightarrow C = e^{-1}$

$x(t) = -e^{-t} + e^{-1}t + 1$

Control Theory

$\dot{x} = u$

$H = 2xe^{-t} - 2xu - u^2 + pu$

$\frac{\partial H}{\partial u} = -2x - 2u + p \Rightarrow u = \frac{p}{2} - x$

$\dot{p} = -\frac{\partial H}{\partial x} = -(2e^{-t} - 2u) = 2\dot{x} - 2e^{-t}$

$\dot{x} = u = \frac{p}{2} - x$

$\ddot{x} = \dot{u} = \frac{\dot{p}}{2} - \dot{x}$

$\ddot{x} = \dot{x} - e^{-t} - \cancel{\dot{x}} = -e^{-t}$

$\dot{x} = e^{-t} + C$

$x = -e^{-t} + Ct + D$

same as above

2. max $\int_0^2 \left(3 - x^2 - 2\dot{x}^2\right) dt \quad x(0) = 1 \quad x(2) \geq 4$

CoV

$\frac{\partial F}{\partial x} = -2x \qquad \frac{\partial F}{\partial \dot{x}} = -4\dot{x} \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -4\ddot{x}$

$-2x + 4\ddot{x} = 0$

$\ddot{x} - \frac{1}{2}x = 0$

$r^2 - \frac{1}{2} = 0$

$r = \pm \frac{1}{\sqrt{2}}$

$x(t) = Ae^{\frac{t}{\sqrt{2}}} + Be^{\frac{-t}{\sqrt{2}}}$

$x(0) = A + B = 1 \Rightarrow B = 1 - A$

If $x(2) > 4$, $\frac{\partial F}{\partial \dot{x}}\Big|_{t=2} = 0$

$\dot{x} = \frac{A}{\sqrt{2}} e^{\frac{t}{\sqrt{2}}} - \frac{1-A}{\sqrt{2}} e^{\frac{-t}{\sqrt{2}}}$

$\frac{\partial F}{\partial \dot{x}}\Big|_{t=2} = \frac{-4A}{\sqrt{2}} e^{\sqrt{2}} - \frac{4-4A}{\sqrt{2}} e^{-\sqrt{2}} = 0$

$\frac{-4A}{\sqrt{2}} e^{\sqrt{2}} + \frac{4A}{\sqrt{2}} e^{-\sqrt{2}} = \frac{4}{\sqrt{2}} e^{-\sqrt{2}}$

$A = \frac{2\sqrt{2}e^{-\sqrt{2}}}{2\sqrt{2}(e^{-\sqrt{2}} - e^{\sqrt{2}})} = 1 - e^{-2\sqrt{2}} \qquad B = e^{-2\sqrt{2}}$

With A, B as such $x(2) < 4 \Rightarrow$ contradiction

$x(2) = 4$

$Ae^{\frac{2}{\sqrt{2}}} + (1-A)e^{\frac{-2}{\sqrt{2}}} = 4$

$$\boxed{\begin{aligned} A = 1 - B &= \frac{4e^{\sqrt{2}} - 1}{e^{2\sqrt{2}} - 1} \\ x(t) &= Ae^{\frac{t}{\sqrt{2}}} + Be^{\frac{-t}{\sqrt{2}}} \end{aligned}}$$

F concave in $x, \dot{x}$ so this is solution

Control Theory

$H = 3 - x^2 - 2u^2 + pu \qquad \dot{x} = u$

$\frac{\partial H}{\partial u} = -4u + p \Rightarrow u^* = \frac{p}{4}$

$\dot{p} = -\frac{\partial H}{\partial x} = -(-2x) = 2x$

$\ddot{p} = 2\dot{x} = 2u = \frac{p}{2} \Rightarrow \ddot{p} - \frac{1}{2}p \Rightarrow r = \pm\frac{1}{\sqrt{2}}$

$Ae^{\frac{t}{\sqrt{2}}} + Be^{\frac{-t}{\sqrt{2}}}$ the rest is the same as above

J. $\int_0^1 (-2\dot{x} - \dot{x}^2) e^{\frac{-t}{10}} dt \qquad x(0) = 1 \qquad x(1) = 0$

CoV

$\frac{\partial F}{\partial x} = 0 \qquad \frac{\partial F}{\partial \dot{x}} = (-2 - 2\dot{x}) e^{\frac{-t}{10}}$

$(-2 - 2\dot{x}) e^{\frac{-t}{10}} = k$

$\dot{x} = k e^{\frac{t}{10}} - 1$

$x = k e^{\frac{t}{10}} - t + C$

$x(0) = k + C = 1$

$x(1) = (1 - C) e^{\frac{1}{10}} - 1 + C = 0$

$C = 1 \quad k = 0$

$\boxed{x^* = -t + 1}$

F concave in $x, \dot{x}$ so solution is optimal

Control theory

$H = (-2u - u^2) e^{\frac{-t}{10}} \qquad u = \dot{x}$

$\frac{\partial H}{\partial u} = (-2 - 2u) e^{\frac{-t}{10}} \Longrightarrow u^* = -1$

$\dot{p} = \frac{-\partial H}{\partial x} = 0$

$u = \dot{x} = -1$

$x = -t + C$

$\boxed{x^* = -t + 1}$